Managing Inventory for a Multi-divisional Corporation with Cash Pooling

Problem definition: Cash pooling is a prevalent and powerful management tool that allows each division's cash balance to be transferred to a single account managed by the corporate treasury. We examine the benefit of cash pooling on inventory replenishment for a corporation with multiple divisions, each replenishing inventory to meet its local demand while receiving cash payments from customers. The corporation can invest excess cash in assets to receive a return or finance inventory internally by selling the invested assets. There are transaction costs for exchanging assets. The objective is to obtain the optimal joint cash and inventory policy that maximizes the expected net worth (equity) of the corporation at the end of a finite-time horizon.

Academic/Practical Relevance: While the reported benefits of cash pooling in the finance literature are mainly associated with the reduction of financing costs, the value of cash pooling is not clear from a perspective of improving operational efficiency. We fill the gap in this paper. The considered problem is practically relevant as it is concerned with working capital management and academically relevant as we relax the no-transaction-cost assumption in Modigliani and Miller (1958) and model the cash flow dynamics generated by the operational decision.

Methodology: We formulate this problem into a dynamic program, and show that the problem is equivalent to minimizing the expected total costs, consisting of the cash-related costs and the inventory-related costs.

Results: Due to curse of dimensionality, we provide a simple and effective heuristic derived from the construction of an innovative lower bound to the optimal value function. Our lower bound improves the so-called Lagrangian-relaxation bound and the induced-penalty bound in the literature. Our solution approach can be applied to the classic one-warehouse-multi-retailer inventory system with non-stationary demands.

Managerial Implications: Our study provides guidance on a firm's pooling strategy. When demands are increasing and positively correlated (stationary and negatively correlated, respectively) between the divisions, cash (inventory, respectively) pooling yields significant value whereas inventory (cash, respectively) pooling yields marginal value. A firm should implement a full integration strategy by pooling both cash and inventory when the demands are increasing and negatively correlated. In addition, the benefit of cash pooling on reducing the inventory-related costs often outweighs that of reducing the cash-related costs, suggesting that cash pooling is a powerful tool to reduce mismatches between demand and supply.

Key words: multi-divisional firm, cash pooling, inventory management, working capital

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1. Introduction

Working capital management is concerned with a firm's short-term liquidity. Maintaining a healthy working capital requires a firm to strike a balance between cash reserved for operational activities and cash invested in assets for returns or growth. The biggest drain on a firm's short-term cash flow is inventory. That is, a firm often experiences cash shortage during the slow seasons or the growth stage of products; see Grasby et al. (2007) for a company example. The former occurs because of cash being trapped in the unsold inventory whereas the latter occurs because of insufficient funds to stock inventory for the surging demand. Instead of short-term loans, which may be costly or difficult to obtain during economic downturns, firms with products sold by separate business affiliates or in dispersed geographical locations may consolidate cash within the entire organization through a centralized treasury system. This is the so-called cash pooling strategy, which allows divisions to borrow from one another to cover temporary deficits, and eliminates interest payments on the short-term debt for financing inventory. In fact, the cash pooling strategy, a powerful tool of internal financing, has been successfully implemented in many multi-national, multi-divisional corporations, including Roche, Tyco International, and Lenovo, through in-house banks or third-party financial services platforms (Zhang et al. 2012, Zhang et al. 2011, Zhang et al. 2012).

While the benefits of cash pooling have been reported in business cases and studied in the finance literature, e.g., lower financing costs than external loans, reduction of interest paid, and improving liquidity, they are centered around financial and managerial benefits. The value of cash pooling on improving operational efficiency has not yet been examined because there is sparse research that models the cash flow dynamics generated by the operational decision. Interestingly, several consulting firms, such as BearingPoint, J.P. Morgan, and McKinsey, have noticed the need of looking beyond the traditional finance-centric working capital practices and advocated a full collaboration between procurement, logistics, and finance departments when managing working capital decisions (BearingPoint 2011, Grasby et al. 2007, J.P. Morgan 2015). In this paper, we aim to fill this gap by providing a model of such joint decision in the inventory context under cash pooling.

We consider a firm with multiple divisions in dispersed locations. Each division replenishes inventory from its supplier to meet the demand in each period. The demands between periods are independent but not necessarily identical, and demands between divisions may be correlated. The firm's financial flows are centrally managed by the corporate treasury, which has a master account that nets the cash balances between divisions in each period. Specifically, a division receives cash payment after a customer order arrives and fulfills as much of the order as possible. Backorders are accumulated; inventory holding and backorder costs are calculated in each period. The cash payment is transferred directly to the master account held by the corporate treasury, which in turn, pays the ordered inventory to the suppliers. The firm needs to decide how much cash to reserve for purchasing inventory so that the rest can be used for investing in assets (e.g., financial markets, facility expansion, R&D, etc.) with a positive return. The value of the invested assets and the accumulated returns are recorded in an investment account. The firm may sell the invested assets with transaction costs if it needs to increase the cash holding (e.g., purchasing additional inventory for an increasing demand). The objective is to obtain the optimal joint inventory replenishment and cash reservation policy such that the expected net worth (equity) is maximized at the end of the horizon.

We formulate this problem into a dynamic program and show that maximizing the expected net worth is equivalent to minimizing the total expected costs (= inventory related costs plus cash related costs). We first characterize the optimal cash retention policy – the firm monitors the system working capital (i.e., cash and monetary inventory value in the system) at the beginning of each period and maintains the cash holding within an interval determined by two thresholds. However, due to curse of dimensionality, it is not possible to fully characterize the joint optimal policy. Thus, from a perspective of implementation and revealing insights, we aim to derive a simple and effective heuristic policy. In order to evaluate the performance of heuristic policies, we construct a lower bound to the optimal cost. For the inventory model with a linear allocation constraint, two lower bounds have appeared in the literature, i.e., the induced-penalty bound for the i.i.d. demand model (Chen and Zheng 1994) and the Lagrangian-relaxation bound for the finite-horizon model (Goel and Gutierrez 2011). Using a novel idea of introducing a parameter that adjusts the cash holding amount in each period, we show that our lower bound dominates the above two known ones. We refer to the parameter as the *cash-holding multiplier*. In fact, our lower bound converges to the optimal value at the expense of computational efforts. More importantly, we are able to derive a heuristic policy based on the lower bound functions. The heuristic policy has a simple structure: the cash retention policy has exactly the same threshold structure as that of the optimal one; the inventory policy for the division is a modified base-stock policy (i.e., ordering up to a level determined by cash available). The exact heuristic policy parameters require an input of effective cash-holding multipliers. We propose two methods: the static policy assumes that the cash-holding multipliers are constants across all time periods; the dynamic policy solves the best multipliers dynamically according to the system state in each period. A numerical study suggests that the dynamic policy outperforms the static policy with a small margin, suggesting the importance of choosing an effective initial cash-holding multiplier. We note that our approach of constructing the lower bound and the heuristic can be applied to the classic one-warehouse-multi-retailer inventory system with non-stationary demands. As shown in Appendix A, the resulting lower bound outperforms those in the multi-echelon literature. Finally, one can view our model as a retailer who manages multiple products with a cash constraint. Thus, our results can comfortably be applied to such a setting.

Our contributions are threefold. First, from an intellectual perspective, we incorporate cash flows into the classic inventory model by relaxing the no-transaction-cost assumption in the Modigliani and Miller (1958) theorem. We propose a new objective of maximizing the expected net worth (equity), which generalizes the traditional cost-minimization objective in the inventory models. We also advocate the importance of monitoring a new system state, i.e., system working capital – an extension of echelon concept in the inventory literature – for the optimal joint decision. Second, on the technical side, we characterize the optimal system working capital policy, and develop an effective heuristic based on an innovative lower bound to the optimal value function. The solution approach can be applied to classic multi-echelon distribution systems. Third, in terms of managerial insights, our study provides guidance on a firm's cash and inventory pooling strategies. While pooling is always beneficial to firms, it often requires a costly investment in the infrastructure. Our study suggests that a firm should implement a cash pooling strategy when the demands are increasing and positively correlated between the divisions because inventory pooling yields marginal value in this case. On the other hand, when the demands are stationary and negatively correlated, a firm should implement an inventory pooling strategy. A firm can obtain the maximum benefit with a full integration strategy by pooling both cash and inventory when the demands are increasing and negatively correlated. Finally, while cash pooling reduces both the inventory-related costs and the cash-related costs, the cost reduction of the former often exceeds that of the latter, suggesting that the cash-pooling practice is a powerful tool to reduce mismatches between demand and supply.

The rest of the paper is organized as follows. §2 reviews the relevant literature. We highlight the differences between our model and those in the literature. §3 describes the model in detail. §4 characterizes the properties of the optimal cash management policy. §5 constructs a novel lower bound on the optimal cost. §6 introduces simple and effective heuristic policies based on the optimality analysis and the lower bound functions. §7 presents numerical results. §8 concludes our work and discusses some extensions. Appendix A demonstrates how our methodology can be applied to a multi-echelon distribution system and generates a new lower bound that outperforms those in the literature. Appendix B shows all proofs.

2. Literature Review

There are primarily five streams of research related to our work: cash management, capacitated inventory systems, multi-echelon distribution systems, integrated cash and inventory models, and investment and consumption models in the economics literature. From a modeling perspective, this paper is a generalization of a serial supply chain with multiple divisions studied in Luo and Shang (2015) to a distributed supply chain. Thus, to save space, we only review papers most relevant to our model and refer the reader to Luo and Shang (2015) for a complete review on the cash management and capacitated inventory literature, as well as the other related papers.

The considered problem and the classic distribution system share some similarities. In particular, the cash level held at the corporate treasury is similar to the inventory amount held at the warehouse, and both models require an allocation of resources to the downstream locations. For this reason, we first review multi-echelon distribution (or one-warehouse-multi-retailer) systems. In the seminal work of Clark and Scarf (1960), the authors point out that an optimal policy, if it exists, would be very difficult to compute and implement due to curse of dimensionality. Nevertheless, under the so-called balance assumption (i.e., inventory can be instantaneously transferred between the downstream locations), they show that an echelon base-stock policy is optimal. Federgruen and Zipkin (1984) extend this result to an infinite-horizon problem. They show that the resulting echelon base-stock levels are stationary and provide a simple algorithm to compute the base-stock levels. Chen and Zheng (1994) consider the i.i.d. demand model with fixed order costs in each location. They streamline and simplify the optimality proof of Clark and Scarf, and construct two lower bounds based on innovative cost allocation schemes. Given that the optimal policy is difficult to obtain, researchers instead focus on easy and implementable policies. The research work in this category typically provides the optimization algorithms for given policies or develops heuristics to facilitate the implementation. For the systems without fixed order costs, base-stock policies are often considered, e.g., Sherbrooke (1968), Graves (1985), Axsäter (1990), and Gallego et al. (2007). We refer the reader to Simchi-Levi and Zhao (2012) and Shang (2011) for an extensive review. The existing results mainly focus on the system with i.i.d. demands.

Our work differs from those in the literature in three aspects. First, most papers for the distribution system focus on the infinite-horizon model with given stationary policies. Our model, on the other hand, considers the finite horizon with nonstationary and correlated demands and we aim to obtain the optimal policy. Second, unlike the inventory level resumed by replenishment at the warehouse in the distribution system, the cash level in our model is resumed not only by the cash retention decision, but also by the random sales received from the customers. In addition, the cash level and the inventory value together define an important state variable, i.e., system working capital, which influences the optimal joint decision. Lastly, unlike the linear ordering costs in the distribution system, following Baumol (1952), we assume that linear transaction costs incur when the cash is transferred between the master account and the investment account. These piece-wise linear transaction costs make the cash retention policy more complicated.

We next review papers that incorporate financial flows into inventory systems. Most of these papers are based on single-stage systems, see Buzacott and Zhang (2004), Chao et al. (2008), Babich (2010), Yang and Birge (2011), Tanrisever et al. (2012), Li et al. (2013), and references therein. A few papers study the joint operation and financial decisions in serial inventory systems. Hu and Sobel (2007) consider a two-stage model with financial constraints. The objective is to maximize the expected dividends in a finite horizon. They show that an echelon base-stock policy is no longer optimal. Song et al. (2014) introduce an accounting framework to study the impact of different payment times on the resulting system cost. Protopappa-Sieke and Seifert (2010) consider a two-stage supply chain and reveal qualitative insights on the allocation of working capital between the supply chain partners via a simulation study. The motivation of this work is most related to Luo and Shang (2015). They consider a serial system that integrates cash flows into material flows whereas we study a distribution system. Nevertheless, due to the cash allocation, our analysis is very different from that of Luo and Shang. The only paper to our knowledge that incorporates financial flows into a distribution system is Chou et al. (2013). They consider a distribution network in which trade credit contracts are employed between the supplier and retailers who face deterministic demands. They show that the supplier who receives a long trade credit term from its external vendor may not provide a long trade credit term to its retailers. Their research question as well as the model setup are quite different from ours.

Finally, we notice that our model is related to the investment and consumption models in the economics literature (e.g., Constantinides 1979, Shreve and Soner 1994, Liu 2004, Kallsen and Muhle-Karbe 2017 and references therein). Most of these papers attempt to investigate the optimal investment and consumption decisions with multiple risky assets in the presence of transaction costs. The decision of transferring cash between the master account and investment account in our model is similar to the models in this literature. The main difference is that, instead of using consumption as a decision variable to deplete cash, our model considers inventory ordering, which involves complicated system dynamics that lead to random cash flows.

3. Model and Problem Formulation

We consider a centralized-control supply chain in which a firm manages the inventory and cash flows for its N divisions over T periods. Without loss of generality, for simplicity, we set $N = 2$ for the subsequent discussions. In each period, each division i reviews its inventory level (= on-hand inventory - backorders) and pipeline inventory, and places an order to an outside supplier with ample stock to satisfy the local demand D_i. Demands are stochastic and independent between periods but not necessarily identical. The demands between divisions may be correlated. There are no inventory transshipments between the divisions. (For example, the divisions may be located in dispersed regions so transshipment is not economically feasible.) The replenishment lead time for division i, denoted by L_i, is a positive constant. Unsatisfied demands are fully backlogged.

To better manage cash, the firm implements a centralized treasury system with cash pooling; that is, the firm creates a master account that consolidates cash flows related to operational activities (i.e., inventory payments paid to the suppliers and collected from customers) for the entire supply chain. Specifically, at the beginning of each period, after receiving customer's payments from the previous period, the firm decides an amount of cash kept in the master account used for inventory replenishment for the current period. The remaining cash will be used for investments, i.e., purchasing a portfolio of assets recorded in an investment account. We assume that the investment yields a return rate η in each period. On the other hand, the firm may finance inventory internally by selling the invested assets if necessary. We refer to the determination of cash amount for inventory replenishment as the *cash retention decision*. The amount of available cash in the master account determines the total inventory amount that can be ordered by both divisions. We consider the so-called pay-on-order scheme, i.e., a payment is created when an order is placed or demand arrives. We assume that all cash transfers are instantaneous. Figure 1 shows the inventory and cash flows in the system. The circle represents the master account, and the oval represents the investment account. The divisions are denoted by rectangles, and they order from outside suppliers which are represented by triangles. Inventory and cash flows are denoted by solid and dash arrows, respectively.

The cash retention decision under the two-account setting reflects the practice. A firm typically does not hold unnecessary cash for operations as it loses the potential benefit from external investments. On the other hand, selling the invested assets for cash to assist operations usually is less costly than external short-term financing.[1] Thus, we assume that the firm finances inventory

[1] The Pecking Order Theory or Pecking Order Model (Myers and Majluf 1984) states that the cost of financing increases as companies use sources of funds where the degree of asymmetric information is higher. As companies raise more and more capital, it becomes increasingly hard to obtain such funding internally. Instead, they are forced to resort to bank debt and public equity. These sources of funding tend to be more expensive.

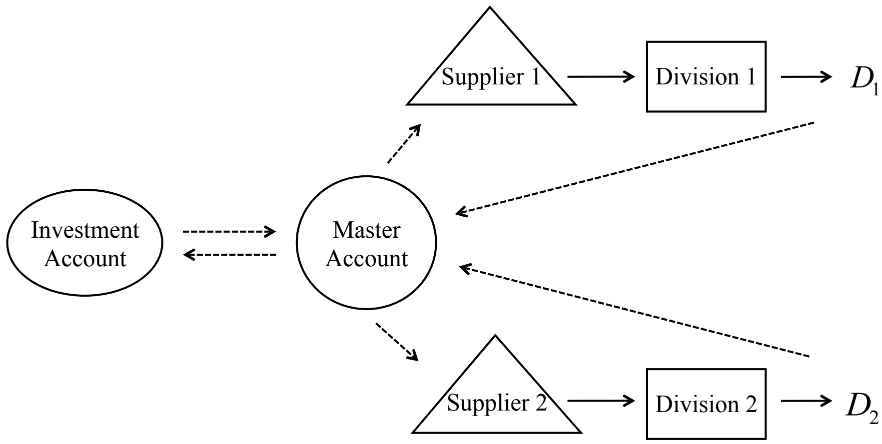

Figure 1 **Inventory and cash flows for a two-divisional firm.**

internally before making short-term loans externally.

There are unit transaction costs β_I and β_O for purchasing and selling assets, respectively. Note that the invested assets in our model can be either short-term, such as interest gains from the money markets, or long-term, such as fixed assets. In general, the transaction cost is higher when the invested assets are less liquid (Miller 1965).[2] Our modeling approach has flexibility to describe different investment activities. In fact, this two-account setting is consistent with the cash management literature (e.g., Baumol 1952, Tobin 1956, and Miller and Orr 1966), which does not consider the detailed inventory dynamics. More importantly, the existence of transaction costs represents a financial friction, which relaxes one assumption made in the Modigliani-Miller theorem (1958). This relaxation makes the considered model of joint operational and financial flows relevant.[3] We assume $0 \le \beta_O < \eta < \beta_O + \alpha\beta_I$, where α is a discount rate. (As we shall see later, $\alpha = 1/(1+\eta)$.) If the second inequality does not hold, the corporation will never invest in the external portfolio. Similarly, if the third inequality fails, the corporation will never hold any cash in the master account. Note that β_O is often relatively small in practice as transferring cash for external investments within a firm causes minimal costs.

Define c_i and p_i as the unit ordering cost and selling price of the product for division i, respectively. We assume that $c_i(1+\eta) < p_i$, for $i = 1, 2$, which means the unit profit is higher than the

[2] Examples of transaction costs are broker's fees, opportunity losses of selling assets, the cost of finding buyers and sellers, bargaining costs, administration costs.

[3] When the financial markets are perfect, i.e., no taxes, no transaction costs, no bankruptcy costs, no information asymmetry, and equivalence of borrowing costs for both companies and investors (borrowing rate equals return rate), the MM theory states that a firm's value is determined by its earning power and by the risk of its underlying assets (capital structure), and this value is independent of the way it chooses to finance its investments. This is the so-called the capital structure irrelevance principle. It can be shown that when both β_I and β_O are zero, our model reduces to a classic inventory problem with two divisions replenishing inventory to fulfill their own demand.

return from the external investment; otherwise, the corporation would never invest in inventory. For division i, there is a physical holding cost h_i for each unit of inventory held in each period and a physical backorder cost b_i for each unit of backorders in each period. Here, the physical holding cost rate refers to the costs related to inventory storage, insurances, shrinkage, etc., which does not include the financial opportunity cost due to holding inventory. The physical backorder cost rate should be viewed as the same way – it is the tangible, monetary penalty costs related to backlogging, e.g., expediting delivery costs. The objective of the corporation is to obtain the optimal joint cash retention and inventory replenishment policy such that the expected net worth (i.e., equity) at the end of the horizon is maximized. The corporation's net worth is equal to the sum of the value of the investment assets, the cash balance in the master account, and the total inventory value.[4] Notice that we do not consider the long-term assets and the liabilities. Thus, maximizing the expected net worth is the same as maximizing the expected working capital assumed in Luo and Shang (2018). As shown later, this objective is also equivalent to minimizing the total cash- and inventory-related costs assumed in Luo and Shang (2015).

The sequence of events in a period is summarized below (see Figure 2). At the beginning of the period, (1) division i receives the order placed L_i periods ago. After reviewing the inventory status of the two divisions and the cash balance in the master account, (2) the firm makes the cash retention decision and places an order to the outside vendor for each division; the payments to the vendors are immediately deducted from the master account. During the period, demands are realized and sales revenue is transferred to the master account. At the end of the period, the physical inventory holding and backorder costs and cash transaction costs are calculated and deducted from the master account.

For periods $t = 1, 2, ..., T$, and divisions $i = 1, 2$, we define the following variables:

$\widehat{x}_{i,t}$ = inventory level at division i after event (1);

$z_{i,t}$ = order quantity for division i in event (2);

$\mathbf{q}_{i,t} = (q^1_{i,t}, q^2_{i,t}, ..., q^{(L_i-1)}_{i,t})$, the pipeline inventory for division i after event (1), where $q^\tau_{i,t}$ is the units to be delivered in τ periods, $\tau = 1, 2, ..., L_i - 1$;

$\widehat{w}_t$ = cash balance in the master account before event (2);

I_t = gross value of external investments before event (2);

[4] We assume that there are no trade credit contracts (accounts receivable and payable) between the firm and the suppliers. Having trade credit terms, such as net terms, only changes the timing of cash flows, but does not affect the essence of the problem.

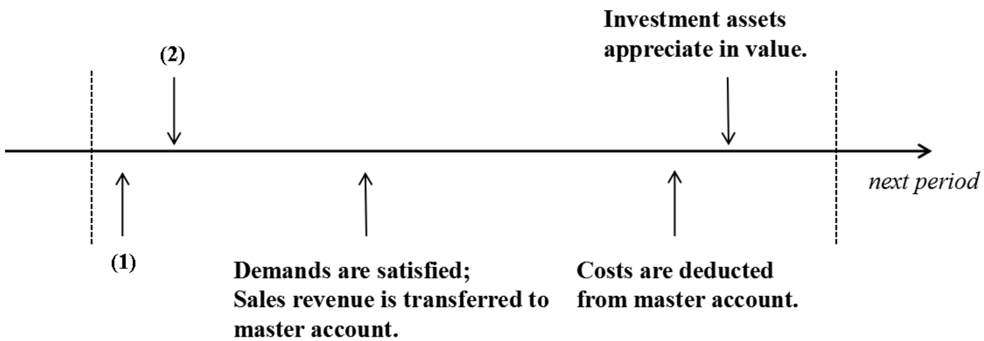

Figure 2 **The events timeline in one period.**

W_t = net worth of the corporation before event (2);

v_t = cash amount transferred from the external investment to the master account in event (2).

Note that the decision variable v_t could be positive or negative. In particular, v_t^+ represents the cash transferred into the master account, whereas v_t^- is the cash transferred to the external investment account (we define $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$). The total cash transaction cost in period t can be calculated by $F(v_t) = \beta_I v_t^+ + \beta_O v_t^-$. The net worth W_t is the sum of the gross value of the external investment assets, the cash balance in the master account, and the total inventory value, i.e., $W_t = I_t + \widehat{w}_t + \sum_{i=1}^2 c_i(\widehat{x}_{i,t} + \mathbf{1}^T \mathbf{q}_{i,t})$, where $\mathbf{1}$ represents a column vector with all elements equal to one.

We define $\widehat{H}_i(x) = h_i x^+ + b_i x^-$, which represents the physical inventory holding and backorder costs at division i in period t given the end-of-period inventory level x. The system dynamics can be characterized as follows: for $i = 1, 2$,

$$\widehat{x}_{i,t+1} = \widehat{x}_{i,t} + q_{i,t}^1 - D_{1,t}, \tag{1}$$

$$\mathbf{q}_{i,t+1} = (q_{i,t}^2, ..., q_{i,t}^{(L_i-1)}, z_{i,t}), \tag{2}$$

$$\widehat{w}_{t+1} = \widehat{w}_t + v_t - \sum_{i=1}^2 (c_i z_{i,t}) + \sum_{i=1}^2 (p_i D_{i,t}) - F(v_t) - \sum_{i=1}^2 \widehat{H}_i(\widehat{x}_{i,t} - D_{i,t}), \tag{3}$$

$$I_{t+1} = (1+\eta)(I_t - v_t). \tag{4}$$

Equations (1) and (2) describe the inventory dynamics. As shown in (3), the transaction cost and physical inventory holding and backorder costs are deducted from the master account. Equation (4) describes that the value of external investments is increased by $(1+\eta)$.[5]

[5] Equation (4) implies that the firm can borrow externally with the borrowing rate η if the investment account is negative. This assumption is standard in the cash management literature; see, for example, Baumol (1952), p.525. This is a rare event that does not affect the essence of the model – A comprehensive simulation study with different

We first consider the constraints on the inventory and cash decisions. The total inventory value that can be ordered by both divisions should be less than the cash balance in the master account after the cash retention decision, i.e., $\widehat{w}_t + v_t - c_1 z_{1,t} - c_2 z_{2,t} \geq 0$. Thus, the feasible decision set $\widehat{S}_t(\widehat{w}_t)$ in period t can be expressed as

$$\widehat{S}_t(\widehat{w}_t) = \{v_t, z_{1,t}, z_{2,t} |\ z_{1,t} \geq 0,\ z_{2,t} \geq 0,\ (\widehat{w}_t + v_t) \geq (c_1 z_{1,t} + c_2 z_{2,t})\}.$$

We next turn to the objective function. From Equations (1) - (4), the net worth in period $t+1$ is

$$\begin{aligned} W_{t+1} &= I_{t+1} + \widehat{w}_{t+1} + \sum_{i=1}^{2} c_i(\widehat{x}_{i,t+1} + \mathbf{1}^T \mathbf{q}_{i,t+1}) \\ &= (1+\eta)W_t + \sum_{i=1}^{2} [(p_i - c_i)D_{i,t}] \\ &\quad -\eta[\widehat{w}_t + v_t + \sum_{i=1}^{2} c_i(\widehat{x}_{i,t} + \mathbf{1}^T \mathbf{q}_{i,t})] - F(v_t) - \sum_{i=1}^{2} \widehat{H}_i(\widehat{x}_{i,t} - D_{i,t}). \end{aligned} \tag{5}$$

Thus, by applying Equation (5) recursively, the corporation's end-of-horizon net worth can be written as

$$\begin{aligned} W_{T+1} &= (1+\eta)^T W_1 + \sum_{t=1}^{T} (1+\eta)^{T-t} \Big[\sum_{i=1}^{2} [(p_i - c_i)D_{i,t}] \\ &\quad -\eta[\widehat{w}_t + v_t + \sum_{i=1}^{2} c_i(\widehat{x}_{i,t} + \mathbf{1}^T \mathbf{q}_{i,t})] - F(v_t) - \sum_{i=1}^{2} \widehat{H}_i(\widehat{x}_{i,t} - D_{i,t}) \Big]. \end{aligned} \tag{6}$$

Note that the expected revenue $\sum_{i=1}^{2} [(p_i - c_i)D_{i,t}]$ is a constant and W_1 is the initial net worth. Thus, maximizing the expected end-of-horizon net worth $\mathbb{E}[W_{T+1}]$ is equivalent to minimizing the expected total inventory and cash related costs over the whole horizon. Specifically, the original problem is equivalent to

$$\begin{aligned} &\min_{\substack{(v_t, z_{1,t}, z_{2,t}) \in \widehat{S}_t(\widehat{w}_t); \\ t=1,\ldots,T}} \mathbb{E}\Big[\sum_{t=1}^{T} (1+\eta)^{T-t} \Big[\eta[\widehat{w}_t + v_t + \sum_{i=1}^{2} c_i(\widehat{x}_{i,t} + \mathbf{1}^T \mathbf{q}_{i,t})] + F(v_t) + \sum_{i=1}^{2} \widehat{H}_i(\widehat{x}_{i,t} - D_{i,t}) \Big] \Big] \\ &= (1+\eta)^{T-1} \min_{\substack{(v_t, z_{1,t}, z_{2,t}) \in \widehat{S}_t(\widehat{w}_t); \\ t=1,\ldots,T}} \mathbb{E}\Big[\sum_{t=1}^{T} \alpha^{t-1} \Big[\eta[\widehat{w}_t + v_t + \sum_{i=1}^{2} c_i(\widehat{x}_{i,t} + \mathbf{1}^T \mathbf{q}_{i,t})] \end{aligned}$$

demand forms suggests that the chance of external financing in a period is less than 1.2%. We made this assumption to facilitate our analysis. Note that the financial friction in our model is the transaction cost, not the external borrowing cost. Nevertheless, the short-term borrowing model and the current model are related – a special case of our model by fixing the cash retention policy as $v_t = (c_1 z_1 + c_2 z_2 - \hat{w}_t)$ reduces to a short-term borrowing model with a borrowing rate of $(R_t + \beta_O)$ and a return rate of $(R_t - \beta_I)$, where R_t is a compound interest rate in period t based on η. The proof and simulation study are available upon request from the authors.

$$+ F(v_t) + \sum_{i=1}^{2} \widehat{H}_i(\widehat{x}_{i,t} - D_{i,t})\Big]\Big], \qquad (7)$$

where $\alpha = 1/(1+\eta)$, which can be interpreted as a discount rate.

The problem in (7) can be solved by the dynamic program shown below. Define $\widehat{V}_t(\widehat{w}_t, \widehat{\mathbf{x}}_t, \mathbf{q}_{1,t}, \mathbf{q}_{2,t})$ as the minimum expected total costs from period t to T over all feasible decisions, where $\widehat{\mathbf{x}}_t = (x_{1,t}, x_{2,t})$. The optimality recursion is

$$\widehat{V}_t(\widehat{w}_t, \widehat{\mathbf{x}}_t, \mathbf{q}_{1,t}, \mathbf{q}_{2,t}) = \min_{(v_t, z_{1,t}, z_{2,t}) \in \widehat{S}_t(\widehat{w}_t)} \left\{ \widehat{G}_t(\widehat{w}_t, \widehat{\mathbf{x}}_t, \mathbf{q}_{1,t}, \mathbf{q}_{2,t}, v_t) + \alpha \mathbb{E}\left[\widehat{V}_{t+1}(\widehat{w}_{t+1}, \widehat{\mathbf{x}}_{t+1}, \mathbf{q}_{1,t+1}, \mathbf{q}_{2,t+1})\right] \right\}, \qquad (8)$$

where

$$\widehat{G}_t(\widehat{w}_t, \widehat{\mathbf{x}}_t, \mathbf{q}_{1,t}, \mathbf{q}_{2,t}, v_t) = \eta[\widehat{w}_t + v_t + \sum_{i=1}^{2} c_i(\widehat{x}_{i,t} + \mathbf{1}^T \mathbf{q}_{i,t})] + F(v_t) + \sum_{i=1}^{2} \mathbb{E}[\widehat{H}_i(\widehat{x}_{i,t} - D_{i,t})], \qquad (9)$$

with $\widehat{V}_{T+1}(\cdot) = 0$, and $\widehat{\mathbf{x}}_{t+1}, \mathbf{q}_{1,t+1}, \mathbf{q}_{2,t+1}$, and $\widehat{w}_{t+1}$ following the dynamics in (1)-(3).

The minimum cost formulation provides a clear economic explanation. The first term on the right-hand side of (9) can be viewed as the opportunity cost of holding cash and inventory which incurs a potential profit loss from stable capital appreciation (see Allen and Hafer 1984; Luo and Shang 2015). The second term is the transaction cost for cash transfers between the master account and the investment account. The third term is the total expected physical inventory holding and backorder cost at both divisions in period t. Note that the single-period cost function in (9) does not include the inventory ordering cost $\mathbf{c}^T \mathbf{z}_t$. This is because the total working capital is not affected by inventory procurement: the increased inventory value is equal to the decreased cash amount in the master account.

Simplified Model and Echelon Formulation

It is difficult to obtain the optimal policy for the problem in (8) as it is a multi-dimensional dynamic program subject to curse of dimensionality. One immediate idea is to follow the approach of Clark and Scarf (1960) who defined echelon terms to reduce the dimension. However, our problem is more complicated as the cash transition in Equation (3) involves non-linear terms $F(v_t)$ and $\sum_{i=1}^{2} \widehat{H}_i(\widehat{x}_{i,t} - D_{i,t})$. To proceed, we propose a simplified model in which these two non-linear terms are omitted from the cash dynamics. (See Luo and Shang (2018) and Luo and Shang (2015) for the same treatment on a single-stage and serial model, respectively.)[6] Consequently, the cash dynamics in (3) become

[6] We compare the terminal system net worth of the simplified model and that of the exact model under the optimal policy for two-divisional firms. For the 240 representative instances, the average percentage difference is 0.94%, suggesting that the simplified model is a good approximation to the exact model. Running the simulation for the systems with more than two divisions yields a similar result.

$$\widehat{w}_{t+1} = \widehat{w}_t + v_t - \mathbf{c}^T\mathbf{z}_t + \mathbf{p}^T\mathbf{D}_t. \tag{10}$$

Notice that $F(v_t)$ and $\sum_{i=1}^{2}\widehat{H}_i(\widehat{x}_{i,t} - D_{i,t})$ are still recorded in the objective function.

In this simplified model, the inventory order in period t does not affect the inventory level and cash flow until it arrives at the division. It allows us to use the idea of echelon transformation to simplify the formulation. Define $x_{i,t}$ as the inventory position at division i, and w_t as the cash balance in the master account plus the total inventory value. We refer to w_t as the *system working capital*.

For period $t = 1,2,...,T+1$, and division $i = 1,2$, we have

$$x_{i,t} = \widehat{x}_{i,t} + \mathbf{1}^T\mathbf{q}_{i,t}, \quad \text{and} \quad w_t = \widehat{w}_t + \sum_{i=1}^{2} c_i(\widehat{x}_{i,t} + \mathbf{1}^T\mathbf{q}_{i,t}).$$

Accordingly, we also define the following echelon variables:

$$y_{i,t} = x_{i,t} + z_{i,t},\ i = 1,2, \quad \text{and} \quad r_t = w_t + v_t,$$

where $y_{i,t}$ represents the order-up-to inventory position at division i, and r_t is the corporation's working capital after the cash retention decision.

Under these echelon variables, the system dynamics become

$$x_{i,t+1} = y_{i,t} - D_{i,t},\ i = 1,2, \quad \text{and} \quad w_{t+1} = r_t + (\mathbf{p} - \mathbf{c})^T\mathbf{D}_t,$$

and the feasible decision set becomes

$$S_t(\mathbf{x}_t) = \left\{r_t, \mathbf{y}_t | \mathbf{y}_t \geq \mathbf{x}_t,\ r_t \geq \mathbf{c}^T\mathbf{y}_t\right\}, \tag{11}$$

where $r_t \geq \mathbf{c}^T\mathbf{y}_t$ is the new cash *allocation constraint* that requires the cash used for inventory replenishment cannot exceed the cash balance in the master account.

The simplified problem can be reformulated as

$$V_t(w_t, \mathbf{x}_t) = \min_{(r_t, \mathbf{y}_t) \in S_t(\mathbf{x}_t)} \left\{ G_t(w_t, r_t, \mathbf{y}_t) + \alpha \mathbb{E}\Big[V_{t+1}(r_t + (\mathbf{p} - \mathbf{c})^T\mathbf{D}_t, \mathbf{y}_t - \mathbf{D}_t)\Big] \right\}, \tag{12}$$

where

$$G_t(w_t, r_t, \mathbf{y}_t) = \eta r_t + F(r_t - w_t) + \sum_{i=1}^{2} H_{i,t}(y_{i,t}), \tag{13}$$

$$H_{i,t}(y_{i,t}) = \alpha^{L_i}\mathbb{E}[\widehat{H}_i(y_{i,t} - \sum_{j=0}^{L_i} D_{i,t+j})]. \tag{14}$$

The corresponding terminal function is $V_{T+1}(w_{T+1}, \mathbf{x}_{T+1}) = 0$. We refer to (12) as the echelon formulation of the simplified model. Our subsequent analysis will be imposed on the problem (12).

4. Optimality Analysis

Unlike Luo and Shang (2015), the simplified problem still suffers from the issue of curse of dimensionality as there is a working capital allocation problem for the corporate treasury. This is the same issue as the inventory allocation problem in the multi-echelon distribution system. Thus, our objective here is to derive a simple and effective heuristic. To that end, we first explore some properties for the optimal joint policy in this section. These optimality analysis will motivate us to develop a novel lower bound on the optimal cost as well as the heuristic policy.

The following lemma shows that the optimal value function is jointly convex. All proofs can be found in Appendix B.

Lemma 1. *The function $V_t(w_t, \mathbf{x}_t)$ is jointly convex in $(w_t, \mathbf{x}_t)$ for all t.*

With Lemma 1, we form the following KKT conditions for problem (12) to investigate the optimal solution.

$$\eta + \beta_I + \alpha \frac{\partial}{\partial r_t} \mathbb{E}[V_{t+1}(r_t + (\mathbf{p}-\mathbf{c})^T \mathbf{D}_t, \mathbf{y}_t - \mathbf{D}_t)] - \lambda_t = 0, \quad \text{if } r_t > w_t; \tag{15}$$

$$\eta - \beta_O + \alpha \frac{\partial}{\partial r_t} \mathbb{E}[V_{t+1}(r_t + (\mathbf{p}-\mathbf{c})^T \mathbf{D}_t, \mathbf{y}_t - \mathbf{D}_t)] - \lambda_t = 0, \quad \text{if } r_t < w_t; \tag{16}$$

$$\eta - \beta_O + \alpha \frac{\partial}{\partial r_t} \mathbb{E}[V_{t+1}(r_t + (\mathbf{p}-\mathbf{c})^T \mathbf{D}_t, \mathbf{y}_t - \mathbf{D}_t)] < \lambda_t$$

$$< \eta + \beta_I + \alpha \frac{\partial}{\partial r_t} \mathbb{E}[V_{t+1}(r_t + (\mathbf{p}-\mathbf{c})^T \mathbf{D}_t, \mathbf{y}_t - \mathbf{D}_t)], \quad \text{if } r_t = w_t; \tag{17}$$

$$\frac{\mathrm{d}}{\mathrm{d}y_{i,t}} H_{i,t}(y_{i,t}) + \alpha \frac{\partial}{\partial y_{i,t}} \mathbb{E}[V_{t+1}(r_t + (\mathbf{p}-\mathbf{c})^T \mathbf{D}_t, \mathbf{y}_t - \mathbf{D}_t)] - \mu_{i,t} + c_i \lambda_t = 0, \quad \text{for } i = 1, 2; \tag{18}$$

$$\lambda_t (r_t - \mathbf{c}^T \mathbf{y}_t) = 0; \tag{19}$$

$$\mu_{i,t}(y_{i,t} - x_{i,t}) = 0, \quad \text{for } i = 1, 2. \tag{20}$$

The first four equations represent the first-order conditions for decision variables r_t and $\mathbf{y}_t$. The last two equations are the complementary slackness conditions, where λ_t and $\mu_{i,t}$ are the nonnegative Lagrange multipliers associated with constraints $r_t \geq \mathbf{c}^T \mathbf{y}_t$ and $y_{i,t} \geq x_{i,t}$, respectively. We next explore the following bounds on the first-order partial derivatives of the objective functions $V_t(w_t, \mathbf{x}_t)$, which will be useful in deriving further results.

Lemma 2. *For any period t, the first-order partial derivative of the optimal value function V_t to the working capital w_t is bounded by $-\beta_I \leq \frac{\partial}{\partial w_t} V_t(w_t, \mathbf{x}_t) \leq \beta_O$, for all $(w_t, \mathbf{x}_t)$.*

Lemma 2 indicates that the marginal optimal cost of the working capital is bounded below by $-\beta_I$ and above by β_O. To see why this marginal optimal cost is bounded above by β_O, imagine that we have achieved the system optimal cost with given $(w_t, \mathbf{x}_t)$. For w_t increased by one unit, this implies that cash increases by one unit as $\mathbf{x}_t$ is fixed. With this additional cash, one strategy, which may not be optimal, is to invest externally, resulting in a transaction cost β_O. Thus, the marginal optimal cost to this additional cash must be smaller than β_O, as one can always perform no worse than simply investing externally. On the other hand, the maximum optimal system cost that can be saved from this additional cash is that it had been kept in the master account, instead of transferring from the investment account. Thus, the maximum cost saving is β_I.

Let r_t^* represent the optimal working capital level after the cash retention decision and $y_{i,t}^*$ be the optimal order-up-to inventory position at division i. Let $(\lambda_t^*, \boldsymbol{\mu}_t^*)$ be the optimal Lagrangian multipliers. As a consequence of conditions (15)-(20) and Lemma 2, we can obtain the structural properties of $(\lambda_t^*, \boldsymbol{\mu}_t^*)$ summarized in the following theorem.

THEOREM 1. *For any period t, the optimal working capital r_t^* and the optimal values of Lagrangian multipliers $(\lambda_t^*, \boldsymbol{\mu}_t^*)$ are related as follows:*

(i) *If $r_t^* > w_t$, then $\eta + \beta_I - \alpha\beta_I \le \lambda_t^* \le \eta + \beta_I + \alpha\beta_O$ and $\prod_{i=1}^{2} \mu_{i,t}^* = 0$.*

(ii) *If $r_t^* = w_t$, then $0 \le \lambda_t^* \le \eta + \beta_I + \alpha\beta_O$ and $\prod_{i=1}^{2} \mu_{i,t}^* \ge 0$.*

(iii) *If $r_t^* < w_t$, then $0 \le \lambda_t^* \le \eta - \beta_O + \alpha\beta_O$ and $\prod_{i=1}^{2} \mu_{i,t}^* \ge 0$.*

Theorem 1 shows three possible optimal solutions that depend on the system working capital level. Part (i) describes a scenario in which cash is not sufficient and additional cash is transferred from the investment account to the master account for inventory replenishment. From Lemma 2, it is clear that $\lambda_t^* \ge \eta + \beta_I - \alpha\beta_I > 0$. This implies that the working capital constraint is binding, i.e., $(r_t^* = c_1 y_{1,t}^* + c_2 y_{2,t}^*)$. In other words, if cash is transferred into the master account, all of the transferred cash has to be used in purchasing inventory, making $\prod_{i=1}^{2} \mu_{i,t}^* = 0$ (at least one division places an order). Note that λ_t^* is the shadow price, which means how much cost can be reduced if the system has one unit of "free" cash to order. In such case, the system can avoid the cash holding cost η, the transaction cost β_I, and the potential cost of disposing it for investment in the next period, i.e., $\alpha\beta_O$. This explains the right-hand side of the bound for λ_t^*. The left-hand side can be explained similarly: $(\beta_I - \alpha\beta_I)$ represents the actual transferring cost reduction if the cash is transferred this period instead of the following period.

Part (ii) describes a scenario in which the optimal system working capital level after the cash retention decision is the same as the initial working capital level in period t. This implies that no cash is transferred. In such case, both divisions may not necessarily order in period t so $\prod_{i=1}^{2} \mu_{i,t}^* \ge 0$. The right-hand size of the bound for λ_t^* has the same economic meaning as that of Part (i). As for the left-hand side, if the system has a right amount of working capital, it is possible that the additional free cash does not bring any benefit so the left-hand side bound is zero. Lastly, Part (iii) describes a scenario in which there is sufficient cash, and the excess cash is transferred from the master account to the outside investment account. In this case, similar to Part (ii), both divisions may not necessarily order in period t. As for the right-hand side of the bound for λ_t^*, consider the cost of keeping one additional unit of cash. The system incurs one unit of the cash holding cost but transfers one unit less of cash to the investment account. In addition, this unit potentially has to be disposed in the following period. Thus, the net cost of this additional unit is $(\eta - \beta_O + \alpha\beta_O)$. In other words, if the firm has one free unit of cash, the net benefit would be the net cost shown above.

In summary, Theorem 1 suggests that the optimal cash retention policy is a two-threshold policy. If the working capital is too low, cash should be transferred into the master account up to a lower-threshold level. In this case, all of the transferred cash should be used for inventory ordering. The benefit of transferring cash in this case is to avoid a significant backorder cost (at the expense of incurring the transaction cost β_I). On the other hand, if the working capital is too high, cash should be disposed to the external investment account until an upper-threshold level. The benefit of transferring cash externally is to avoid the cash holding cost (at the expense of the transaction cost β_O). These properties are useful to develop our heuristic in the subsequent section.

While it is difficult to obtain the optimal joint policy, we are able to characterize the exact one for the system with i.i.d. demands. Consider the case in which the divisions independently manage the inventory and cash flows and the cash generated from sales is always sufficient for inventory replenishment (the divisions therefore transfer all excess cash to external investments). As such, division i faces the following dynamic program:

$$V_{i,t}(x_t) = \min_{y_t \geq x_t} \left\{ (\eta - \beta_O + \alpha\beta_O)c_i y_t + H_{i,t}(y_t) + \alpha\mathbb{E}[V_{i,t+1}(y_t - D_{i,t})] \right\}, \tag{21}$$

where $V_{i,T+1}(x_{T+1}) = 0$.

The unit cost $(\eta - \beta_O + \alpha\beta_O)$ reflects the opportunity cost of inventory replenishment. On the one hand, ordering one more unit of inventory incurs a potential profit loss from stable capital appreciation (i.e., η). On the other hand, as the division transfers all excess cash to external investments, ordering one more unit of inventory reduces the cash disposal cost in the current period (i.e., $-\beta_O$). However, this additional inventory unit will be sold and bring in additional cash in the next period, it will make the division dispose more cash from the master account in the next period (i.e., $\alpha\beta_O$).

In the stationary setting, it is well known that the optimal policy for problem (21) is a myopic base-stock policy with reorder points S_i defined by

$$S_i = \arg\min \left\{ (\eta - \beta_O + \alpha\beta_O) c_i y_t + H_{i,t}(y_t) \right\}.$$

The following proposition demonstrates the optimal inventory and cash policy in the stationary setting under some mild conditions.

Proposition 1. *For the system with i.i.d. demands, if the initial working capital satisfies* $w_1 \geq (c_1 S_1 + c_2 S_2)$ *and the terminal function is* $V_{T+1}(w_{T+1}, \mathbf{x}_{T+1}) = \beta_O w_{T+1}$, *the optimal inventory policy is a base-stock policy with the reorder point* S_i *and the firm transfers excess cash to the investment account after inventory payment.*

Whenever a customer orders, the corporation will receive sufficient funds for the inventory replenishment in the following period because $p_i > c_i$. Thus, the cash allocation is no longer a concern as the corporation always has sufficient cash to fulfill the inventory order.

5. Lower Bound

As the optimal policy is difficult to characterize, we aim to propose a simple and effective heuristic. This section develops a tight lower bound to the optimal cost in order to evaluate the effectiveness of a heuristic policy. It turns out that this lower bound will lead to an effective heuristic presented in §6.

An innovative idea of constructing the lower bound is to introduce a linear function $a_t(r_t - \mathbf{c}^T\mathbf{y}_t)$, where $(r_t - \mathbf{c}^T\mathbf{y}_t)$ is the cash amount in the master account after inventory replenishment, and a_t is nonnegative and can be considered as an incentive for the corporate treasury to hold one more unit of cash in the master account. For this reason, we shall refer to a_t as the *cash-holding multiplier*. By incorporating this additional term to our problem (12), we can construct an auxiliary system as follows:

$$\underline{V}_t(w_t, \mathbf{x}_t | \mathbf{a}_t) = \min_{\substack{r_t \geq \mathbf{c}^T\mathbf{y}_t \\ \mathbf{y}_t \geq \mathbf{x}_t}} \left\{ \underline{G}_t(w_t, r_t, \mathbf{y}_t | a_t) + \alpha \mathbb{E}\Big[\underline{V}_{t+1}(r_t + (\mathbf{p} - \mathbf{c})^T \mathbf{D}_t, \mathbf{y}_t - \mathbf{D}_t | \mathbf{a}_{t+1}) \Big] \right\}, \qquad (22)$$

where

$$\underline{G}_t(w_t, r_t, \mathbf{y}_t | a_t) = \eta r_t + F(r_t - w_t) + \sum_{i=1}^{2} H_{i,t}(y_{i,t}) \overbrace{-a_t(r_t - \mathbf{c}^T\mathbf{y}_t)}^{\text{savings due to holding cash}}.$$

Here, $\mathbf{a}_t$ is a $(T - t + 1)$-dimensional vector $(a_t, a_{t+1}, ..., a_T)$. Clearly, compared with the original system, the auxiliary system has exactly the same cost function except the additional savings term.

Given a nonnegative a_t, the problem in (22) is clearly a lower bound on the optimal one, i.e., $\underline{V}_t(w_t, \mathbf{x}_t|\mathbf{a}_t) \leq V_t(w_t, \mathbf{x}_t)$. Moreover, the original system is a special case of the auxiliary system with $\mathbf{a}_t = \mathbf{0}$.

We next derive a lower bound on the optimal cost of the auxiliary system. The derivation is based on Clark and Scarf (1960) and illustrated in Chen and Zheng (1994). We generalize it to our model. Imagine that each inventory unit at the division i is composed of a cash-equivalent component 0 and a division-specific component i. The cash-equivalent component 0 is distributed through the corporate treasury to each division. The component i is replenished directly from division i's supplier. For a given $\mathbf{a}_t$, let $\underline{J}_{i,t}(y_t|\mathbf{a}_t)$ denote total cost of division i in period t when its inventory position after ordering is y_t and the optimal inventory decisions are employed for period $t+1, ..., T$, i.e.,

$$\underline{J}_{i,t}(y_t|\mathbf{a}_t) = H_{i,t}(y_t) + a_t c_i y_t + \alpha \mathbb{E}[\underline{V}_{i,t+1}(y_t - D_{i,t}|\mathbf{a}_{t+1})]. \tag{23}$$

The function $\underline{J}_{i,t}(y_t|\mathbf{a}_t)$ is convex in y_t. Let

$$S_{i,t}(\mathbf{a}_t) = \arg\min_{y_t \in \mathbb{R}} \underline{J}_{i,t}(y_t|\mathbf{a}_t). \tag{24}$$

The function $\underline{J}_{i,t}(y_t|\mathbf{a}_t)$ can be separated into two parts. The first part, $\underline{\Gamma}_{i,t}(y_t|\mathbf{a}_t)$ defined in (26), is the cost resulted from an insufficient amount of cash-equivalent component 0 (so that division i cannot order up to the desired level $S_{i,t}(\mathbf{a}_t)$). The second part is the remaining of $\underline{J}_{i,t}(y_t|\mathbf{a}_t)$. Now, we assign $\underline{\Gamma}_{i,t}(y_t|\mathbf{a}_t)$ to the corporate treasury, as it is the cost caused by insufficient cash. Let the optimal cost for the corporate treasury in period t under such a cost allocation scheme be $\underline{V}_{H,t}(\cdot|\mathbf{a}_t)$. These cost functions are shown below: for $i = 1, 2$,

$$\begin{aligned}\underline{V}_{H,t}(w_t|\mathbf{a}_t) &= \min_{r_t \geq \mathbf{c}^T \mathbf{y}_t} \Big\{ (\eta - a_t) r_t + F(r_t - w_t) + \sum_{i=1}^{2} \underline{\Gamma}_{i,t}(y_{i,t}|\mathbf{a}_t) \\ &\qquad + \alpha \mathbb{E}[\underline{V}_{H,t+1}(r_t + (\mathbf{p} - \mathbf{c})^T \mathbf{D}_t|\mathbf{a}_{t+1})] \Big\}, \end{aligned} \tag{25}$$

$$\underline{V}_{i,t}(x_{i,t}|\mathbf{a}_t) = \min_{y_t \geq x_{i,t}} \Big\{ H_{i,t}(y_t) + a_t c_i y_t - \underline{\Gamma}_{i,t}(y_t|\mathbf{a}_t) + \alpha \mathbb{E}[\underline{V}_{i,t+1}(y_t - D_{i,t}|\mathbf{a}_{t+1})] \Big\}, \tag{26}$$

where

$$\underline{\Gamma}_{i,t}(y_t|\mathbf{a}_t) = \begin{cases} \underline{J}_{i,t}(y_t|\mathbf{a}_t) - \underline{J}_{i,t}(S_{i,t}(\mathbf{a}_t)|\mathbf{a}_t), & \text{if } \; y_t < S_{i,t}(\mathbf{a}_t); \\ 0, & \text{otherwise.} \end{cases} \tag{27}$$

With this cost allocation scheme, we have decoupled the total optimal cost in period t into three separate ones; $\underline{V}_{i,t}(\cdot|\mathbf{a}_t)$ is the optimal cost for division i whereas $\underline{V}_{H,t}(\cdot|\mathbf{a}_t)$ is the optimal cost for the corporate treasury. As the three subsystems do not need to be coordinated due to the decoupling, the resulting sum of the optimal costs is a lower bound to that of the auxiliary system, which, in turn, a lower bound to that of the original system. Theorem 2 summarizes the result.

Theorem 2. *For all t and $(w_t, \mathbf{x}_t)$, $V_t(w_t, \mathbf{x}_t) \geq \underline{V}_t(w_t, \mathbf{x}_t|\mathbf{a}_t) \geq \underline{V}_{H,t}(w_t|\mathbf{a}_t) + \sum_{i=1}^{2} \underline{V}_{i,t}(x_{i,t}|\mathbf{a}_t)$.*

It is worth noting that when $\mathbf{a}_t = \mathbf{0}$, the term $a_t(r_t - \mathbf{c}^T \mathbf{y}_t)$ becomes zero, and the above cost decomposition scheme is degenerated to that of Chen and Zheng's induced-penalty bound. To obtained an improved lower bound than Chen and Zheng's induced-penalty bound, we can search for $\mathbf{a}_1$ such that the sum of the optimal costs of three subsystems is maximized:

$$\max_{\mathbf{a}_1} \quad \left\{ \underline{V}_{H,1}(w_1|\mathbf{a}_1) + \sum_{i=1}^{2} \underline{V}_{i,1}(x_{i,1}|\mathbf{a}_1) \right\}. \tag{28}$$

We first explain how to obtain the optimal cost for each subsystem with fixed $\mathbf{a}_1$. $\underline{V}_{i,1}(x_{i,1}|\mathbf{a}_1)$ is obtained by solving a single-stage inventory problem; $\underline{V}_{H,1}(w_1|\mathbf{a}_1)$ is obtained by using an algorithm similar to the one solving the classic distribution system under the so-called *balance assumption* (see Clark and Scarf 1960, Federgruen and Zipkin 1984). Let $u_t(\mathbf{a}_t)$ and $l_t(\mathbf{a}_t)$ denote the thresholds of optimal working capital levels, where

$$u_t(\mathbf{a}_t) = \sup\left\{ r_t : \frac{\mathrm{d}\underline{J}_{H,t}(r_t|\mathbf{a}_t)}{\mathrm{d}r_t} \leq \beta_O \right\}, \tag{29}$$

$$l_t(\mathbf{a}_t) = \sup\left\{ r_t : \frac{\mathrm{d}\underline{J}_{H,t}(r_t|\mathbf{a}_t)}{\mathrm{d}r_t} \leq -\beta_I \right\}, \text{ and} \tag{30}$$

$$\underline{J}_{H,t}(r_t|\mathbf{a}_t) = (\eta - a_t) r_t + \min_{\mathbf{c}^T \mathbf{y}_t \leq r_t} \sum_{i=1}^{2} \underline{\Gamma}_{i,t}(y_{i,t}|\mathbf{a}_t) + \alpha \mathbb{E}[\underline{V}_{H,t+1}(r_t + (\mathbf{p} - \mathbf{c})^T \mathbf{D}_t | \mathbf{a}_{t+1})]. \tag{31}$$

We shall use these two thresholds to control the cash flow. Theorem 3 summarizes the optimal policy for each subsystem with fixed $\mathbf{a}_t$.

Theorem 3.

(i) The optimal inventory policy for division i is a base-stock policy with the base-stock level $S_{i,t}(\mathbf{a}_t)$, which is nonincreasing in the value of a_τ for all time periods $\tau \geq t$.

(ii) The cash retention policy for the corporate treasury is to maintain the cash-equivalent component level between $l_t(\mathbf{a}_t)$ and $u_t(\mathbf{a}_t)$.

We now turn to solving $\mathbf{a}_1$ in (28). To this end, we propose an idea of identifying the search region for $\mathbf{a}_1$ by connecting our lower bound to the Lagrangian relaxation of the simplified model. Denote by $\boldsymbol{\lambda}_t = (\lambda_t, \lambda_{t+1}, ..., \lambda_T)$ the Lagrange multipliers associated with the cash allocation constraints over periods t to T. The Lagrange relaxation of the simplified problem is expressed as

$$V_t^L(w_t, \mathbf{x}_t|\boldsymbol{\lambda}_t) = \min_{\substack{r_t \in \mathbb{R} \\ \mathbf{y}_t \geq \mathbf{x}_t}} \left\{ \eta r_t + F(r_t - w_t) + \sum_{i=1}^{2} H_{i,t}(y_{i,t}) - \lambda_t (r_t - \mathbf{c}^T \mathbf{y}_t) \right.$$

$$+\alpha\mathbb{E}[V^L_{t+1}(r_t+(\mathbf{p}-\mathbf{c})^T\mathbf{D}_t,\mathbf{y}_t-\mathbf{D}_t|\boldsymbol{\lambda}_{t+1})]\Big\} \tag{32}$$

with zero terminal value. Note that the difference between the Lagrangian relaxation and the auxiliary system in (22) is the constraint set: Unlike the auxiliary system, we remove the cash allocation constraint $r_t \geq \mathbf{c}^T\mathbf{y}_t$ from constraint set in Lagrangian relaxation. It is easy to show that the optimal cost of the Lagrangian-relaxation problem is equivalent to the total costs of three subsystems. More specifically, the corporate treasury subsystem

$$V^L_{H,t}(w_t|\boldsymbol{\lambda}_t)=\min_{r_t\in\mathbb{R}}\Big\{(\eta-\lambda_t)r_t+F(r_t-w_t)+\alpha\mathbb{E}[V^L_{H,t+1}(r_t+(\mathbf{p}-\mathbf{c})^T\mathbf{D}_t|\boldsymbol{\lambda}_{t+1})]\Big\}, \tag{33}$$

and division i's cost $V^L_{i,t}(x_{i,t}|\boldsymbol{\lambda}_t)$ is the same as $\underline{V}_{i,t}(x_{i,t}|\mathbf{a}_t)$ in (26) given $\boldsymbol{\lambda}_t=\mathbf{a}_t$.

Compared with the treasury subsystem in our lower bound in (25), the Lagrangian treasury cost defined in (33) has no cash constraint and the induced penalty cost, which suggests that when $\mathbf{a}_t=\boldsymbol{\lambda}_t$, our lower bound is an upper bound to the Lagrangian-relaxation problem. With this result, we can prove the following theorem. Let $\boldsymbol{\lambda}^*_t$ be the dual optimal multipliers of the Lagrangian-relaxation problem. Theorem 4 states that when $\mathbf{a}_t=\boldsymbol{\lambda}^*_t$, our lower bound cost is the optimal cost of the simplified system.

Theorem 4. *For all t and $(w_t,\mathbf{x}_t)$, $V_t(w_t,\mathbf{x}_t)=\underline{V}_{H,t}(w_t|\boldsymbol{\lambda}^*_t)+\sum_{i=1}^2\underline{V}_{i,t}(x_{i,t}|\boldsymbol{\lambda}^*_t)$.*

Theorem 4 motivates us to use the range of Lagrangian multipliers found in Theorem 1 to search for the best $\mathbf{a}_t$. To simplify the calculation, for our lower bound, we set $\mathbf{a}_1=(a_1,a_1,...,a_1)$ for all T and search over $[0,\eta+\beta_I+\alpha\beta_O]$ to find the best a_1. (The lower bound cost can be improved at the expense of computational efforts.) We observe an interesting result: when the system requires more cash holding, a_1 tends to be larger. To explain this observation, note that the lower bound is constructed under the balance assumption. This assumption applied to our model postulates that when the division i cannot order the inventory up to the target level $S_{i,t}(a_t)$ due to cash shortage, the other division can transfer excess inventory to the division i instantaneously. Thus, when we maximize the lower bound over a_1, we in effect aim to lower the effect of the balance assumption, or, equivalently, increasing the cash holding at the corporate treasury.

We use a numerical example to illustrate this result. Figure 3 illustrates the optimal cash-holding multiplier a_1 with respect to the selling price p_i under different demand patterns.[7] As shown, a_1 is decreasing in p_1 for all demand patterns. When p_1 is larger, the firm can gain more profit from

[7] We fix parameters $T=10$, $N=2$, $\eta=0.15$, $\beta_I=1.0$, $\beta_O=0.01$, $h_i=0.25$, $c_i=1$, and the initial states $(w_1,x_{1,1},x_{2,1})=(28,7,7)$.

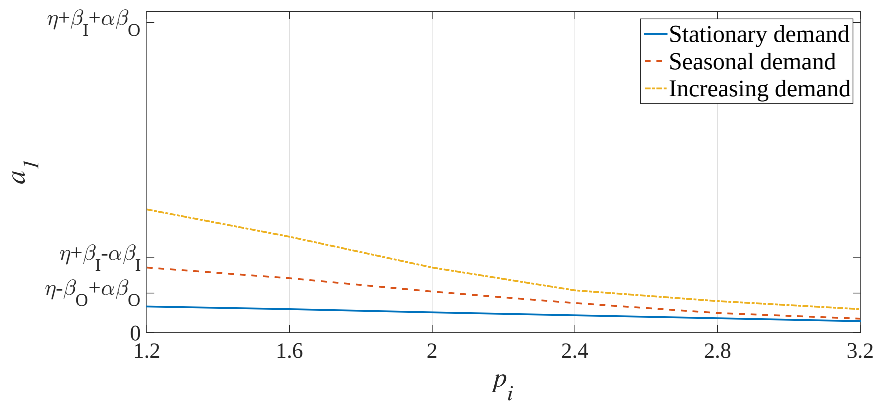

Figure 3 **The optimal a_1 with respect to the selling price p_i under different demand forms.**

product sales. Thus, the division is likely to order up to the desired level through sales and does not need to hold excess cash in the master account, making the cash-holding multiplier a_1 smaller. Notice that under the i.i.d. demand case, a_1 is close to zero for all t and instances. This suggests that Chen and Zheng's induced-penalty bound performs well for the i.i.d. demand case. However, when the demands are non-stationary, the choice of the cash-holding multipliers a_1 becomes an important factor that affects the performance of the lower bound. In particular, we find that a_1 tends to be positive when the demands are increasing.

6. Heuristic

Theorem 3 suggests that the optimal policy for the lower bound system has a simple structure. The firm maintains the system working capital between $l_t(\mathbf{a}_t)$ and $u_t(\mathbf{a}_t)$. If the working capital is lower than the lower threshold, cash is retrieved up to $l_t(\mathbf{a}_t)$; if the working capital is higher than the upper threshold, cash is disposed down to $u_t(\mathbf{a}_t)$. This two-threshold structure is consistent with the optimal cash retention policy in Theorem 1. For the inventory policy, each division simply implements a base-stock policy.

We propose the aforementioned optimal policy of the lower bound system as our heuristic. We need to refine the resulting policy for some rare situations during implementation. First, the cash upper threshold $u_t(\mathbf{a}_t)$ may be lower than the initial inventory value $\mathbf{c}^T\mathbf{x}_t$. As inventory disposal is not allowed, adjusting the working capital down to $u_t(\mathbf{a}_t)$ is not feasible. In our heuristic, if the situation occurs, we just do nothing and keep $r_t^* = \mathbf{c}^T\mathbf{x}_t$. Second, it is possible that the division cannot order up to $S_{i,t}(\mathbf{a}_t)$ due to the cash limitation in the master account. In such case, the division just orders up to the level that minimizes the lower-bound cost determined by the cash allocation constraint.

We now discuss how to choose $\mathbf{a}_t$ in our heuristic. We suggest two methods. For the static policy, the vector $\mathbf{a}_1$ is predetermined at the beginning of the first period and stays unchanged over the entire horizon. This is the same method as we use for the lower bound value defined in (28). For the dynamic heuristic policy, we update the subvector $\mathbf{a}_t$ at the beginning of each period t based on real-time system states. Specifically, in each period t, the corporation solves the following problem:

$$\underset{0\le a_t\le \eta+\beta_I+\alpha\beta_O}{\arg\max}\left\{\underline{V}_{H,t}(w_t|\mathbf{a}_t)+\sum_{i=1}^{2}\underline{V}_{i,t}(x_{i,t}|\mathbf{a}_t)\right\}, \tag{34}$$

where $\mathbf{a}_t=(a_t,a_t,...,a_t)$.

After the selection of $\mathbf{a}_t$, we can generate the corresponding inventory and cash control thresholds and then implement the policy stated by Theorem 3. Clearly, these thresholds depend on the real-time system states $(w_t,\mathbf{x}_t)$.

7. Numerical Study

In this section, we present a set of comprehensive numerical experiments. The goal of this study is threefold. First, we compare our lower bound with the induced-penalty bound developed by Chen and Zheng (1994) for our cash pooling system. (We also compare these two lower bounds for the classic one-warehouse-multi-retailer system in Appendix A). Second, we test the effectiveness of the static and dynamic heuristic policies in §7.2. Third, we reveal insights by quantifying the value of pooling. Our model allows us to study the value of cash pooling and of a fully integrated system in which both cash and inventory are pooled. We shall explain these systems in §7.3. We also investigate how the demand correlation between the divisions affects these values.

We use the following system primitives throughout our numerical study. We set the planning horizon $T=10$ and the number of divisions $N\in\{2,4,8\}$. The following are common parameters between divisions: the replenishment lead time $L_i=1$, the transaction cost $\beta_O=0.01$, the physical holding cost $h_i=0.25$, and the ordering cost $c_i=1$. We change the other six system parameters as follows: $\beta_I\in\{0.2,0.4,0.6\}$, $\eta\in\{0.15,0.2,0.3\}$, $b_i\in\{1,2,3.25\}$, and $p_i\in\{1.4,1.7,2\}$. The demands between divisions are independent, and the demand at division i in period t follows Poisson distribution with mean $\mu_{i,t}$. For each division, we test three demand forms: stationary, increasing and seasonal forms. We set $\mu_{i,t}=5$ for the stationary demand, $\mu_{i,t}=5\times 1.2^{t-1}$ for the increasing demand, and $\mu_i=(5,7,10,7,5,3,1,3,5,7,10)$ for the seasonal demand. Finally, we test a total of nine initial system states $(w_1,\mathbf{x_1})$ shown in Table 1. The total number of instances is 2187.

7.1. Comparison of Lower Bounds

As stated, our lower bound is exactly the optimal cost if the cash-holding multipliers $\mathbf{a}_1$ are equal to the optimal Lagrangian multipliers $\boldsymbol{\lambda}_t^*$. However, obtaining $\boldsymbol{\lambda}_t^*$ is computationally infeasible. Following the logic of the static heuristic, we calculate $\mathbf{a}_1$ according to (28) and report the resulting lower bound cost in our study.

We define the percentage improvement of our lower bound over the induced-penalty bound as

$$\frac{\underline{C}-\underline{C}_{IP}}{\underline{C}_{IP}} \times 100\%,$$

where $\underline{C}$ and $\underline{C}_{IP}$ represent our lower bound and the induced-penalty bound, respectively.

N	$(w_1, x_{i,1}, x_{j,1})$	Stationary		Seasonal		Increasing	
	$i=1,...,N/2; j=N/2+1,...,N$	Avg.	(Max., Std.) %	Avg.	(Max., Std.) %	Avg.	(Max., Std.) %
2	(14, 7, 7)	0.00	(0.00 , 0.00)	1.08	(2.46 , 0.50)	1.25	(2.52 , 0.53)
	(14, 4, 10)	0.00	(0.00 , 0.00)	1.77	(2.54 , 0.42)	2.09	(3.60 , 0.61)
	(14, 1, 13)	0.00	(0.01 , 0.00)	2.69	(4.31 , 0.70)	3.21	(5.43 , 0.81)
4	(28, 7, 7)	0.00	(0.00 , 0.00)	1.24	(2.51 , 0.21)	1.35	(2.69 , 0.36)
	(28, 4, 10)	0.00	(0.01 , 0.00)	1.98	(3.19 , 0.61)	2.20	(3.91 , 0.80)
	(28, 1, 13)	0.00	(0.03 , 0.00)	3.06	(6.54 , 0.64)	3.67	(6.76 , 0.95)
8	(56, 7, 7)	0.00	(0.00 , 0.00)	1.30	(3.27 , 0.32)	1.51	(3.23 , 0.40)
	(56, 4, 10)	0.00	(0.02 , 0.00)	2.30	(4.14 , 0.70)	2.62	(5.80 , 0.84)
	(56, 1, 13)	0.01	(0.05 , 0.01)	3.71	(6.97 , 0.82)	4.42	(7.91 , 1.01)

Table 1 The performance improvement of the lower bound against the induced-penalty bound.

Table 1 summarizes the improvement of our lower bound over the induced-penalty bound. There are three observations. First, the improvement is sensitive to the demand type. For the systems with stationary demand, the improvement of our lower bound is marginal. To explain this observation, recall that the two lower bounds coincide when the incentive factor $\mathbf{a}_1 = \mathbf{0}$. Figure 3 illustrates that the optimal $\mathbf{a}_1$ of problem (28) is close to zero for the stationary systems. It is consistent with Proposition 1 which states that under stationary demands, the cash received in a period is sufficient to pay the ordered inventory so $\mathbf{a}_1$ is close to zero. On the contrary, the improvement is significant for the non-stationary demand cases. For example, for the systems with $N = 8$, the average improvement achieves 4.42% with a maximum of 7.91% for the increasing demand when the initial states are $(56, 1, 13)$. Figure 3 demonstrates that the optimal value of $\mathbf{a}_1$ appears to be much larger than zero under non-stationary demands. Second, the improvement becomes rather significant when the initial inventory levels between the divisions are imbalanced. For example, for the systems with $N = 2$ and increasing demand, the average improvement is about 1.25% when the initial state is $(14, 7, 7)$ (balanced inventory between divisions), whereas the average improvement increases to 3.21% for the initial state $(14, 1, 13)$ (imbalanced inventory between divisions). This observation confirms that reserving more cash in the master account will help coordinate the inventory imbalance between divisions. Third, the improvement becomes larger as the number of divisions increases. This is because when the number of divisions increases, there is a higher probability that the inventory imbalance between the divisions will occur.

N	$(w_1, x_{i,1}, x_{j,1})$	Stationary		Seasonal		Increasing	
	$i=1,...,N/2; j=N/2+1,...,N$	Avg.	(Max., Std.) %	Avg.	(Max., Std.) %	Avg.	(Max., Std.) %
2	(14, 7, 7)	0.07	(0.15 , 0.02)	0.79	(1.56 , 0.25)	1.60	(2.71 , 0.53)
	(14, 4, 10)	0.12	(0.21 , 0.06)	1.59	(2.36 , 0.52)	2.40	(4.01 , 0.51)
	(14, 1, 13)	0.16	(0.32 , 0.04)	2.60	(4.55 , 0.76)	2.93	(5.71 , 0.81)
4	(14, 7, 7)	0.09	(0.19 , 0.02)	1.36	(2.73 , 0.43)	1.82	(3.30 , 0.43)
	(14, 4, 10)	0.14	(0.31 , 0.09)	2.32	(4.21 , 0.62)	2.83	(5.61 , 0.90)
	(14, 1, 13)	0.21	(0.63 , 0.16)	3.48	(6.45 , 1.12)	3.73	(6.81 , 1.30)
8	(14, 7, 7)	0.15	(0.31 , 0.10)	1.72	(3.11 , 0.70)	1.82	(3.45 , 0.71)
	(14, 4, 10)	0.27	(0.51 , 0.12)	3.24	(6.20 , 1.30)	3.69	(7.82 , 1.51)
	(14, 1, 13)	0.33	(0.72 , 0.20)	4.57	(7.70 , 1.23)	4.86	(8.87 , 1.70)

Table 2 The performance of the static policy.

7.2. Heuristic Performance

We test effectiveness of the static and dynamic heuristics by comparing them with our lower bound. The effectiveness of the heuristic is defined as

$$\frac{\overline{C} - \underline{C}}{\underline{C}} \times 100\%,$$

where $\overline{C}$ is the system-wide costs under a certain heuristic and $\underline{C}$ is our lower bound.

We test the parameter combinations in §7.1 with a total $3^7 = 2187$ instances. For each instance, we run a simulation of 1000 iterations to calculate the expected heuristic cost. Tables 2 and 3 present the overall performance of the static heuristic and the dynamic heuristic, respectively. Both heuristic policies perform surprisingly well for the stationary systems (the maximum gap is below 0.7%). For the non-stationary systems, it is conceivable that the heuristics would perform less effectively, and the dynamic heuristic should outperform the static one. The result confirms this conjecture: The average percentage gap is 2.13% for the seasonal demand cases, and 2.85% for the increasing demand cases. Interestingly, the dynamic policy does not significantly improve the performance over the static policy, which suggests that an effective initial cash-holding parameter is crucial. Note that the heuristic is compared with the lower bound cost. Thus, the actual performance would be better than we reported in the table if compared with the optimal cost.

N	$(w_1, x_{i,1}, x_{j,1})$	Stationary		Seasonal		Increasing	
	$i=1,...,N/2; j=N/2+1,...,N$	Avg.	(Max., Std.) %	Avg.	(Max., Std.) %	Avg.	(Max., Std.) %
2	(14, 7, 7)	0.07	(0.13 , 0.02)	0.42	(1.14 , 0.15)	1.21	(2.10 , 0.23)
	(14, 4, 10)	0.10	(0.17 , 0.03)	1.01	(2.14 , 0.27)	1.75	(3.20 , 0.25)
	(14, 1, 13)	0.15	(0.26 , 0.05)	1.89	(3.85 , 0.46)	2.02	(4.29 , 0.41)
4	(14, 7, 7)	0.08	(0.15 , 0.04)	0.91	(1.96 , 0.25)	1.15	(2.71 , 0.53)
	(14, 4, 10)	0.12	(0.26 , 0.09)	1.78	(3.86 , 0.42)	2.19	(4.01 , 0.52)
	(14, 1, 13)	0.16	(0.35 , 0.10)	2.91	(5.85 , 0.61)	3.10	(5.71 , 0.56)
8	(14, 7, 7)	0.10	(0.22 , 0.07)	1.61	(2.96 , 0.40)	1.22	(2.19 , 0.22)
	(14, 4, 10)	0.18	(0.32 , 0.08)	2.69	(5.66 , 0.42)	2.90	(6.01 , 0.50)
	(14, 1, 13)	0.25	(0.47 , 0.14)	3.81	(6.75 , 0.66)	4.03	(7.71 , 0.82)

Table 3 The performance of the dynamic policy.

7.3. Value of Pooling

Our model allows us to study the benefits of cash pooling and a fully integrated system by comparing three systems. For the no-pooling system, each division manages its inventory and cash independently. The cash-pooling system is our current model. The fully integrated system assumes that, in addition to cash pooling, the firm can further pool the division's inventory in a single location that fulfills all demands. The optimal joint cash and inventory policy for a single-location system has been established in Luo and Shang (2015), which can be applied to the no-pooling system and the fully integrated system. Clearly, the fully integrated system performs the best as it consolidates both inventory and cash flows, the cash-pooling system the second, and the no-pooling system the worst. By comparing the cost difference between the cash-pooling system and the fully integrated system, one can justify whether inventory pooling should be implemented as it often requires a significant amount of capital investment in infrastructure.

To quantify the benefits of cash pooling and full integration, we define the following percentages of cost reduction between the above three inventory models.

$$\text{Value of Cash Pooling (VC)} = \frac{C^N - \overline{C}}{C^N} \times 100\%,$$

$$\text{Value of Full Integration (VF)} = \frac{C^N - C^F}{C^N} \times 100\%.$$

where C^N represents the optimal cost of the no-pooling system, C^F the optimal cost of the fully integrated system, and $\overline{C}$ is the total cost under the static heuristic. We define the value of cash pooling (VC) as the percentage of cost reduction from the no-pooling system to the cash pooling system. The value of full integration (VF) is defined similarly. The incremental value of inventory pooling under cash pooling is defined as VI = (VF - VC).

We consider a subset of numerical examples introduced in the beginning of §7. More specifically, we consider cases with $N = 2$ and the initial states $(w_1, x_1, x_2) = (14, 7, 7)$. For the no-pooling system, we consider identical divisions with the same system parameters (i.e., both divisions have the same η, β_I, β_O, etc.) and the same three demand forms specified in §7.1. It is known that the value of pooling is affected by the demand correlation so we consider three demand correlation coefficients between the two divisions as $\rho = \{-0.5, 0, 0.5\}$ in each period. The total number of instances is 729. Table 4 summarizes VF, VC, and VI under different demand forms and correlations.

Table 4 The average value (%) of fully integration, cash pooling and inventory pooling with two divisions.

Demand type	ρ	b_i = 1			2			3.25		
		VF	VC	VI	VF	VC	VI	VF	VC	VI
Stationary	−0.5	4.73	0.72	4.01	7.61	1.11	6.50	10.65	1.77	8.88
	0	1.46	0.53	0.93	2.62	0.69	1.93	3.75	1.14	2.61
	0.5	0.34	0.27	0.07	0.42	0.36	0.06	1.10	0.90	0.20
Seasonal	−0.5	8.56	4.40	4.16	15.18	7.73	7.45	25.08	13.88	11.20
	0	4.39	2.56	1.83	7.90	5.89	2.01	13.20	10.31	2.89
	0.5	2.26	1.30	0.96	5.25	4.01	1.24	9.00	7.10	1.90
Increasing	−0.5	13.96	7.71	6.25	24.65	13.19	11.46	37.05	19.45	17.60
	0	7.38	4.60	2.78	11.76	8.04	3.72	16.28	11.88	4.40
	0.5	3.06	2.01	1.05	5.69	4.09	1.60	9.16	7.17	1.99

Table 5 Pooling strategies under different demand forms and correlations.

	Stationary	Increasing
Positive ρ	No Change	Cash Pooling
Negative ρ	Inventory Pooling	Full Integration

Our numerical study reveals insights on the pooling strategies for firms. When the demands are stationary, we find that the value of cash pooling is generally small under different demand forms and correlations. This is because under the stationary demand, the systems are more likely to have sufficient cash (see Proposition 1) and hence holding cash in the master account does not add much value. On the other hand, the value of inventory pooling is significant (minimal, respectively) when the demands are negatively (positively, respectively) correlated. This result is consistent with that of Eppen (1979).

The value of cash pooling is more profound under the seasonal and increasing demand forms. It is interesting to see that the value of cash pooling is significant but the value of inventory pooling is minimal when the demands are increasing and positively correlated. For example, for the increasing demand instances with $b = 3.25$ and $\rho = 0.5$, the percentage of cost reduction due to cash pooling is 7.17% whereas the cost reduction due to inventory pooling is 1.99%. Finally, the value of cash pooling and the value of inventory pooling are both significant when the demands are increasing and negatively correlated. For the same instances with ρ being -0.5, the percentage cost reduction due to cash pooling is 19.45% whereas the cost reduction due to inventory pooling is 17.60%.

Pooling is always beneficial but it often requires significant investments in infrastructure (e.g., information technology systems, additional logistics costs, etc.). Table 5 provides guidance on a firm's pooling strategy under different demand forms and correlations based on our study. As shown, when demands are stationary (i.i.d.) and positively correlated, it may not be necessary to conduct either inventory pooling or cash pooling as the investment cost may outweigh the benefit due to pooling. On the other hand, when the demands are increasing and positively correlated, the firm should consider implementing cash pooling only, as the inventory pooling may bring little value. Lastly, if the demands are increasing and negatively correlated, the firm can obtain a significant benefit by full integration.

In the finance literature, it is known that cash pooling can reduce financing and transaction costs. To quantify the benefit of cash pooling on reducing inventory-related costs, we divide the total system cost into two parts: the cash-related cost that includes the cash holding and transaction costs, and the inventory-related cost that includes the inventory holding and backorder costs. Specifically, we define

ρ	Demand type: Stationary		Seasonal		Increasing	
−0.5	2.02	0.32	7.20	10.01	11.01	16.51
0	1.34	0.11	4.89	8.01	6.40	11.04
0.5	0.77	0.07	2.96	5.36	3.18	6.51

Table 6 The percentage of cost reduction on the cash-related cost and the inventory-related cost (in solid rectangle) under cash pooling.

$$\text{Cash-related Cost Reduction} = \frac{C^N_{cash} - \overline{C}_{cash}}{C^N_{cash}} \times 100\%,$$
$$\text{Inventory-related Cost Reduction} = \frac{C^N_{inv} - \overline{C}_{inv}}{C^N_{inv}} \times 100\%,$$

where C^N_{cash} and $\overline{C}_{cash}$ are the cash-related costs of no-pooling and cash-pooling systems, respectively, and C^N_{inv} and $\overline{C}_{inv}$ are the inventory-related costs of no-pooling and cash-pooling systems, respectively. For the no-pooling system, we use the optimal cost, whereas for the cash pooling system, we use the cost under the static heuristic.

Table 6 illustrates the cash pooling effect on the cost reduction of the inventory-related cost and cash-related cost. It is interesting to observe that the cost reduction on the inventory-related cost is more significant than that on the cash-related cost when the demands are non-stationary. For example, under the increasing demand with $\rho = -0.5$, the inventory-related cost reduction is about 16.51%, while the cash-related cost reduction is 11.01%. This result suggests that cash pooling reduces not only the cost resulted from transaction costs and financing costs, as suggested in the finance literature, but also the cost of managing inventory by matching supply with demand more efficiently. The latter benefit often outweighs the former.

8. Conclusion

This paper studies a joint inventory and cash management problem for a corporation with multiple divisions. We formulate the problem into a dynamic program and partially characterize the optimal policy. Because of curse of dimensionality, we develop a novel lower bound which is a generalization of the two known lower bounds in the literature. We provide two efficient and simple heuristic policies based on the lower bound functions. We numerically show that the proposed heuristic policies perform near optimally and also examine how system parameters affect the value of cash pooling. We conclude that the value of cash pooling is most significant when the demands are increasing and negatively correlated between the divisions. We also show that cash pooling can effectively alleviate inventory shortage and reduce mismatches of demand and supply. Our model and analysis are comfortably applied to a setting where a cash-constrained retailer replenishes inventory for multiple products. They can also be applied to the classic multi-echelon distribution system under a finite-time horizon with non-stationary demands.

There are two possible extensions of the current work. First, our model and the analysis are based on a centralized control scheme. It is of interest to study a coordination mechanism under which the decisions are decentralized made by the divisions and the corporate treasury. Second, we do not consider a warehouse that can store inventory for both divisions. It is interesting to investigate the benefits of cash pooling and inventory pooling and their relationships (i.e., substitute or complementary) under such model.

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Online Appendix

Managing Inventory for a Multi-divisional Corporation with Cash Pooling

A. Two-Echelon Distribution Systems

Our lower bound can be extended to the traditional distribution inventory system consisting of one warehouse (stage 0) and multiple retailers (stage i, $i=1,...,N$). The retailers replenish their stock from the warehouse, which in turn places orders at an outside supplier with unlimited supply. We define c_0 as the unit ordering cost of the warehouse and set without loss of generality the unit ordering costs of the retailers to be zero. Inventory replenishment at each echelon takes a constant lead time. We assume the material lead time from the outside supplier to warehouse is one period and the lead time from warehouse to retailer i is denoted by L_i. In each period, the warehouse first places an order to the outside supplier if necessary and then distributes the on-hand inventory to the retailers. Left inventory will be carried over to the next period and unsatisfied demands at retailer i are fully backlogged with a unit backorder cost b_i. We denote by h_0 and h_i the unit echelon holding costs at the warehouse and retailer i, respectively. The corporation aims to minimize its total expected costs over T periods.

Let $x_{i,t}$ be the echelon inventory position at stage i in the beginning of period t after receiving shipment and $y_{i,t}$ be order-up-to echelon inventory position. We define the echelon holding/backorder cost associated each echelon as follows:

$$H_{0,t}(y_{0,t}) = \alpha\mathbb{E}[h_0(y_{0,t} - \sum_{i=1}^{N}\sum_{j=0}^{1} D_{i,t+j})],$$

$$H_{i,t}(y_{i,t}) = \alpha^{L_i}\mathbb{E}[h_i(y_{i,t} - \sum_{j=0}^{L_i} D_{i,t+j}) + (b_i + h_0 + h_i)(y_{i,t} - \sum_{j=0}^{L_i} D_{i,t+j})^-], \text{ for } i=1,...,N.$$

Define $\mathbf{x}_t = (x_{1,t}, x_{2,t})$, $\mathbf{y}_t = (y_{1,t}, y_{2,t})$ and $\mathbf{D}_t = (D_{1,t}, D_{2,t})$. The dynamic program is written as

$$V_t(x_{0,t}, \mathbf{x}_t) = \min_{\substack{y_{0,t} \ge x_{0,t} \ge \mathbf{1}^T\mathbf{y}_t \\ \mathbf{y}_t \ge \mathbf{x}_t}} \left\{ c_0(y_{0,t} - x_{0,t}) + H_{0,t}(y_{0,t}) + \sum_{i=1}^{N} H_{i,t}(y_{i,t}) + \alpha\mathbb{E}[V_{t+1}(y_{0,t} - \mathbf{1}^T\mathbf{D}_t, \mathbf{y}_t - \mathbf{D}_t)] \right\}.$$

Incorporating the linear savings due to holding inventory at warehouse $a_t(x_{0,t} - \mathbf{c}^T\mathbf{y}_t)$, the auxiliary system is

$$\underline{V}_t(x_{0,t}, \mathbf{x}_t | \mathbf{a}_t) = \min_{\substack{y_{0,t} \ge x_{0,t} \ge \mathbf{1}^T\mathbf{y}_t \\ \mathbf{y}_t \ge \mathbf{x}_t}} \left\{ c_0(y_{0,t} - x_{0,t}) + H_{0,t}(y_{0,t}) + \sum_{i=1}^{N} H_{i,t}(y_{i,t}) - a_t(x_{0,t} - \mathbf{c}^T\mathbf{y}_t) \right.$$
$$\left. + \alpha\mathbb{E}[\underline{V}_{t+1}(y_{0,t} - \mathbf{1}^T\mathbf{D}_t, \mathbf{y}_t - \mathbf{D}_t | \mathbf{a}_{t+1})] \right\}.$$

Decoupling this system by the cost allocation scheme in Chen and Zheng (1994), our lower bound can be expressed as the sum of following subsystems

$$\underline{V}_{0,t}(x_{0,t} | \mathbf{a}_t) = -(c_0 + a_t)x_{0,t} +$$
$$\min_{y_{0,t} \ge x_{0,t} \ge \mathbf{c}^T\mathbf{y}_t} \left\{ c_0 y_{0,t} + H_{0,t}(y_{0,t}) + \sum_{i=1}^{2} \underline{\Gamma}_{i,t}(y_{i,t} | \mathbf{a}_t) + \alpha\mathbb{E}[\underline{V}_{0,t}(y_{0,t} - \mathbf{1}^T\mathbf{D}_t | \mathbf{a}_{t+1})] \right\},$$

$$\underline{V}_{i,t}(x_{i,t} | \mathbf{a}_t) = \min_{y_t \ge x_{i,t}} \left\{ H_{i,t}(y_t) + a_t c_i y_t - \underline{\Gamma}_{i,t}(y_t | \mathbf{a}_t) + \alpha\mathbb{E}[\underline{V}_{i,t}(y_t - D_{i,t} | \mathbf{a}_{t+1})] \right\}, \text{ for } i=1,...,N.$$

The induced-penalty cost is

$$\underline{\Gamma}_{i,t}(y_t|\mathbf{a}_t) = \begin{cases} \underline{J}_{i,t}(y_t|\mathbf{a}_t) - \underline{J}_{i,t}(S_{i,t}(\mathbf{a}_t)|\mathbf{a}_t), & \text{if } \ y_t < S_{i,t}(\mathbf{a}_t); \\ 0, & \text{otherwise;} \end{cases}$$

where

$$\begin{aligned} \underline{J}_{i,t}(y_t|\mathbf{a}_t) =& H_{i,t}(y_t) + a_t c_i y_t + \alpha \mathbb{E}[\underline{V}_{i,t}(y_t - D_{i,t}|\mathbf{a}_{t+1})], \\ S_{i,t}(\mathbf{a}_t) =& \arg\min_{y_t \in \mathbb{R}} \underline{J}_{i,t}(y_t|\mathbf{a}_t). \end{aligned}$$

Next we conduct a comprehensive numerical study to test the performance of our lower bound for the traditional distribution model. We obtain the value of $\mathbf{a}_1$ through a numerical search

$$\max_{\substack{0 \le a_1 \le \max\{\alpha^{L_i} b_i\} \\ \text{for } \ i=1,2,\ldots,N}} \left\{ \underline{V}_{0,1}(x_{0,1}|\mathbf{a}_1) + \sum_{i=1}^{N} \underline{V}_{i,1}(x_{i,1}|\mathbf{a}_1) \right\}, \tag{A1}$$

where $\mathbf{a}_1 = (a_1, a_1, ..., a_1)$. $\max\{\alpha^{L_i} b_i\}$ for $i = 1, 2, ..., N$ is an upper bound to the shadow price of inventory, which is the maximum possible cost reduced when the system has one more unit "free" inventory held at warehouse. The parameter values are presented in Table A1. We test stationary, seasonal and increasing poisson demands over time periods: $\mu_{i,t} = 5$ for the stationary demand, $\mu_{i,t} = 5 \times 1.2^{t-1}$ for the increasing demand and set $\mu_i = (5, 7, 10, 7, 5, 3, 1, 3, 5, 7, 10)$ for the seasonal demand. To explore the impact of the initial inventory imbalance on the lower bound performance, we set different initial states $(x_{0,1}, \mathbf{x}_1)$. There are totally $3^5 \times 3 \times 3 = 2187$ instances to test. The performance improvement of our lower bound against Chen and Zheng's induced-penalty bound in the distribution system is shown in Table A2. Consistent with observations in the cash pooling system, our lower bound outperforms the induced-penalty bound especially for non-stationary systems.

Table A1 Parameter values for the two-echelon distribution systems.

T	N	α	c	h_0	L_i	h_i	b_i
10	$\{2, 4, 8\}$	$\{0.9, 0.95, 0.99\}$	$\{0.2, 0.4, 0.6\}$	0.2	1	$\{0.1, 0.2, 0.4\}$	$\{2, 3.5, 5\}$

Table A2 The overall performance improvement of our lower bound against the induced-penalty bound for two-echelon distribution systems.

N	$(x_0, x_{i,1}, x_{j,1})$	Stationary		Seasonal		Increasing	
	$i=1,...,N/2; j=N/2+1,...,N$	Avg.	(Max., Std.) %	Avg.	(Max., Std.) %	Avg.	(Max., Std.) %
2	(14, 7, 7)	0.32	(0.55 , 0.16)	0.79	(1.49 , 0.34)	1.36	(3.03 , 0.29)
	(14, 4, 10)	0.72	(1.47 , 0.23)	1.40	(2.69 , 0.40)	2.01	(4.42 , 0.40)
	(14, 1, 13)	1.11	(2.21 , 0.30)	2.31	(4.41 , 0.64)	3.29	(6.50 , 0.51)
4	(14, 7, 7)	0.64	(1.30 , 0.20)	1.08	(2.12 , 0.34)	2.31	(4.50 , 0.50)
	(14, 4, 10)	1.19	(2.41 , 0.35)	1.83	(3.47 , 0.56)	3.24	(6.70 , 0.60)
	(14, 1, 13)	2.09	(4.01 , 0.51)	3.11	(5.90 , 0.60)	4.17	(8.57 , 0.70)
8	(14, 7, 7)	1.51	(3.32 , 0.42)	2.06	(3.81 , 0.50)	3.07	(6.00 , 0.70)
	(14, 4, 10)	2.78	(5.71 , 0.50)	3.25	(5.91 , 0.63)	4.03	(7.41 , 0.82)
	(14, 1, 13)	4.19	(7.16 , 0.65)	4.77	(8.01 , 0.72)	5.18	(8.49 , 1.01)

B. Proofs

Proof of Lemma 1. We prove the result by induction. As $V_{T+1}(w_{T+1}, \mathbf{x}_{T+1}) = 0$, the result trivially holds for period $T+1$. Now, we assume that the result is true for period $t+1$, i.e., $V_{t+1}(w_{t+1}, \mathbf{x}_{t+1})$ is jointly convex in $(w_{t+1}, \mathbf{x}_{t+1})$. We next show that $V_t(w_t, \mathbf{x}_t)$ is also jointly convex in $(w_t, \mathbf{x}_t)$.

As convexity can be preserved by composition with affine functions, $V_{t+1}(r_t + (\mathbf{p}-\mathbf{c})^T \mathbf{D}_t, \mathbf{y}_t - \mathbf{D}_t)$ is jointly convex in $(r_t, \mathbf{y}_t)$. Due to the preservation of convexity under expectation, $\mathbb{E}[V_{t+1}(r_t + (\mathbf{p}-\mathbf{c})^T \mathbf{D}_t, \mathbf{y}_t - \mathbf{D}_t)]$ is also jointly convex in $(r_t, \mathbf{y}_t)$. As $F(x) = \beta_I x^+ + \beta_O x^-$ is a convex function, so is $F(r_t - w_t)$. Consequently, one can readily prove that the single-period cost function $G_t(w_t, r_t, \mathbf{y}_t)$ is joint convex in $(w_t, r_t, \mathbf{y}_t)$. Therefore, the objective function $G_t(w_t, r_t, \mathbf{y}_t) + \mathbb{E}[V_{t+1}(r_t + (\mathbf{p}-\mathbf{c})^T \mathbf{D}_t, \mathbf{y}_t - \mathbf{D}_t)]$ in (12) is jointly convex in $(w_t, r_t, \mathbf{y}_t)$. Note that the constraint set $S_t(\mathbf{x}_t)$ is a convex set. By Proposition 2.2.15 of Simchi-Levi et al. (2004), $V_t(w_t, \mathbf{x}_t)$ is jointly convex in $(w_t, \mathbf{x}_t)$, which completes the induction. □

Proof of Lemma 2. Denote by $(r_t^*, \mathbf{y}_t^*)$ and $(r_{t,\Delta}^*, \mathbf{y}_{t,\Delta}^*)$ the optimal solutions of problem (12) with initial states $(w_t, \mathbf{x}_t)$ and $(w_t + \Delta w, \mathbf{x}_t)$, respectively. Let $J_t(w_t, r_t, \mathbf{y}_t) = G_t(w_t, r_t, \mathbf{y}_t) + \alpha \mathbb{E}[V_{t+1}(r_t + (\mathbf{p} - \mathbf{c})^T \mathbf{D}_t, \mathbf{y}_t - \mathbf{D}_t)]$. To derive the upper bound of $\frac{\partial}{\partial w_t} V_t(w_t, \mathbf{x}_t)$, we show that

$$\begin{aligned}
\frac{V_t(w_t + \Delta w, \mathbf{x}_t) - V_t(w_t, \mathbf{x}_t)}{\Delta w} &= \frac{J_t(w_t + \Delta w, r_{t,\Delta}^*, \mathbf{y}_{t,\Delta}^*) - J_t(w_t, r_t^*, \mathbf{y}_t^*)}{\Delta w} \\
&\le \frac{J_t(w_t + \Delta w, r_t^*, \mathbf{y}_t^*) - J_t(w_t, r_t^*, \mathbf{y}_t^*)}{\Delta w} \\
&= \frac{\beta_I (r_t^* - w_t - \Delta w)^+ + \beta_O (r_t^* - w_t - \Delta w)^-}{\Delta w} \\
&\quad - \frac{\beta_I (r_t^* - w_t)^+ + \beta_O (r_t^* - w_t)^-}{\Delta w} \\
&\le \beta_O,
\end{aligned}$$

where the first inequality follows from the fact that $(r_t^*, \mathbf{y}_t^*)$ is also a feasible solution of problem (12) with the initial state $(w_t + \Delta w, \mathbf{x}_t)$.

Similarly, we can show that

$$\begin{aligned}\frac{V_t(w_t+\Delta w,\mathbf{x}_t)-V_t(w_t,\mathbf{x}_t)}{\Delta w} &= \frac{J_t(w_t+\Delta w, r^*_{t,\Delta},\mathbf{y}^*_{t,\Delta})-J_t(w_t,r^*_t,\mathbf{y}^*_t)}{\Delta w}\\ &\geq \frac{J_t(w_t+\Delta w, r^*_{t,\Delta},\mathbf{y}^*_{t,\Delta})-J_t(w_t,r^*_{t,\Delta},\mathbf{y}^*_{t,\Delta})}{\Delta w}\\ &= \frac{\beta_I(r^*_t-w_t-\Delta w)^+ +\beta_O(r^*_t-w_t-\Delta w)^-}{\Delta w}\\ &\quad -\frac{\beta_I(r^*_t-w_t)^+ +\beta_O(r^*_t-w_t)^-}{\Delta w}\\ &\geq -\beta_I,\end{aligned}$$

where the first inequality follows from the fact that $(r^*_{t,\Delta},\mathbf{y}^*_{t,\Delta})$ is also a feasible solution of problem (12) with the initial state $(w_t,\mathbf{x}_t)$.

Therefore, the results hold. □

Proof of Theorem 1. We prove these results based on the KKT conditions (15)-(20) and Lemma 2.

(i) If $r^*_t > w_t$, the condition (15), together with Lemma 2, implies that

$$\eta+\beta_I-\alpha\beta_I \leq \lambda^*_t = \eta+\beta_I+\alpha\frac{\partial}{\partial r_t}\mathbb{E}[V_{t+1}(r_t+(\mathbf{p}-\mathbf{c})^T\mathbf{D}_t,\mathbf{y}_t-\mathbf{D}_t)] \leq \eta+\beta_I+\alpha\beta_O.$$

It immediately follows that $\lambda^*_t \geq \eta+\beta_I-\alpha\beta_I>0$. The condition (19) implies that $r^*_t=\mathbf{c}^T\mathbf{y}^*_t$. As the initial system working capital is larger than the initial inventory value, i.e., $w_t\geq \mathbf{c}^T\mathbf{x}_t$, $\mathbf{c}^T\mathbf{y}^*_t = r^*_t > w_t \geq \mathbf{c}^T\mathbf{x}_t$. That is, $\mathbf{c}^T\mathbf{y}^*_t > \mathbf{c}^T\mathbf{x}_t$ which implies that either $y_{1,t}>x_{1,t}$ or $y_{2,t}>x_{2,t}$. Consequently, it follows from the condition (20) that either $\mu^*_{1,t}=0$ or $\mu^*_{2,t}=0$, i.e., $\prod_{i=1}^{2}\mu^*_{i,t}=0$. Hence, the result (i) holds.

(ii) If $r^*_t=w_t$, the condition (17) and Lemma 2 imply that $\eta-\beta_O-\alpha\beta_I\leq\lambda^*_t\leq\eta+\beta_I+\alpha\beta_O$. The lower bound is nonpositive due to the assumption that $\eta\leq\beta_O+\alpha\beta_I$. However, the Lagrange multipliers should be nonnegative. As a result, a tighter bound should be $0\leq\lambda^*_t\leq\eta+\beta_I+\alpha\beta_O$. The result $\prod_{i=1}^{2}\mu^*_{i,t}\geq 0$ trivially holds, as the dual optimal multipliers $\mu^*_{i,t}$ are nonnegative.

(iii) Similarly, if $r^*_t<w_t$, it follows from the condition (16) and Lemma 2 that $\eta-\beta_O-\alpha\beta_I\leq\lambda^*_t\leq\eta-\beta_O+\alpha\beta_O$. As the Lagrange multiplier should be nonnegative, the lower bound can be refined to be zero. □

Proof of Proposition 1. We first construct a new system by replacing the cash transaction cost $\beta_I(r_t-w_t)^+ +\beta_O(r_t-w_t)^-$ in the original problem in (12) with $\beta_O(w_t-r_t)$:

$$\widetilde{V}_t(w_t,\mathbf{x}_t)=\min_{(r_t,\mathbf{y}_t)\in S_t(\mathbf{x}_t)}\left\{\widetilde{G}_t(w_t,r_t,\mathbf{y}_t)+\alpha\mathbb{E}\Big[\widetilde{V}_{t+1}(r_t+(\mathbf{p}-\mathbf{c})^T\mathbf{D}_t,\mathbf{y}_t-\mathbf{D}_t)\Big]\right\}, \tag{A2}$$

where $S_t(\mathbf{x}_t)$ is defined in (11) and

$$\widetilde{G}_t(w_t,r_t,\mathbf{y}_t)=\eta r_t+\beta_O(w_t-r_t)+\sum_{i=1}^{2}H_{i,t}(y_{i,t}), \tag{A3}$$

where $H_{i,t}(y_{i,t})$ are defined by (14).

Because $\beta_I(r_t-w_t)^+ +\beta_O(r_t-w_t)^-\geq\beta_O(r_t-w_t)^-\geq\beta_O(w_t-r_t)$, this system is a lower bound to the original system. Specifically, $\widetilde{V}_t(w_t,\mathbf{x}_t)\leq V_t(w_t,\mathbf{x}_t)$ for any $(w_t,\mathbf{x}_t)$. We next analyze the optimal cash retention decision of this system. One can easily prove that $\widetilde{V}_t(w_t,\mathbf{x}_t)$ is increasing in w_t and the marginal cost of cash is bounded above by β_O, i.e., $0\leq\frac{\partial\widetilde{V}_t(w_t,\mathbf{x}_t)}{\partial w_t}\leq\beta_O$. Define $\widetilde{J}_t(w_t,r_t,\mathbf{y}_t)=\widetilde{G}_t(w_t,r_t,\mathbf{y}_t)+\alpha\mathbb{E}\Big[\widetilde{V}_{t+1}(r_t+(\mathbf{p}-\mathbf{c})^T\mathbf{D}_t,\mathbf{y}_t-\mathbf{D}_t)\Big]$. Then, we have

$$\frac{\partial\widetilde{J}_t(w_t,r_t,\mathbf{y}_t)}{\partial r_t}=\eta-\beta_O+\frac{\partial\alpha\mathbb{E}\Big[\widetilde{V}_{t+1}(r_t+(\mathbf{p}-\mathbf{c})^T\mathbf{D}_t,\mathbf{y}_t-\mathbf{D}_t)\Big]}{\partial r_t}$$

$$\geq \eta - \beta_O$$
$$> 0,$$

where the last inequality follows from the assumption $\eta > \beta_O$.

It states that the objective function in (A2) is increasing in r_t. Therefore, the optimal cash retention decision for the new system should satisfy the equation $r_t = \mathbf{c}^T\mathbf{y}_t$. Then we can further simplify the problem in (A2) to

$$\widetilde{V}_t(w_t, \mathbf{x}_t) = \min_{\mathbf{y}_t \geq \mathbf{x}_t} \left\{ \widetilde{G}_t(w_t, \mathbf{y}_t) + \alpha \mathbb{E}\Big[\widetilde{V}_{t+1}(\mathbf{c}^T\mathbf{y}_t + (\mathbf{p}-\mathbf{c})^T\mathbf{D}_t, \mathbf{y}_t - \mathbf{D}_t)\Big] \right\}, \tag{A4}$$

where

$$\widetilde{G}_t(w_t, \mathbf{y}_t) = \eta\mathbf{c}^T\mathbf{y}_t + \beta_O(w_t - \mathbf{c}^T\mathbf{y}_t) + \sum_{i=1}^{2} H_{i,t}(y_{i,t}). \tag{A5}$$

We next show that the problem in (A4) is decomposable. Define the following dynamic program

$$\widetilde{V}_{i,t}(x_t) = \min_{y_t \geq x_t} \left\{ (\eta - \beta_O + \alpha\beta_O)c_i y_t + H_{i,t}(y_t) + \alpha\mathbb{E}[\widetilde{V}_{i,t+1}(y_t - D_{i,t})] \right\}, \tag{A6}$$

where $\widetilde{V}_{i,T+1}(x_{T+1}) = 0$. We now prove by induction that $\widetilde{V}_t(w_t, \mathbf{x}_t)$ can be expressed as $\widetilde{V}_t(w_t, \mathbf{x}_t) = \beta_O w_t + \sum_{i=1}^{2} \widetilde{V}_{i,t}(x_{i,t})$. The result trivially holds for period $T+1$, as $\widetilde{V}_{T+1}(w_{T+1}, \mathbf{x}_{T+1}) = \beta_O w_{T+1}$. Assume that it is true for period $t+1$, i.e., $\widetilde{V}_{t+1}(w_{t+1}, \mathbf{x}_{t+1}) = \beta_O w_{t+1} + \sum_{i=1}^{2} \widetilde{V}_{i,t+1}(x_{i,t+1})$. Then, the problem in (A4) can be rewritten as

$$\begin{aligned}
\widetilde{V}_t(w_t, \mathbf{x}_t) = \min_{\mathbf{y}_t \geq \mathbf{x}_t} & \left\{ \eta\mathbf{c}^T\mathbf{y}_t + \beta_O(w_t - \mathbf{c}^T\mathbf{y}_t) + \sum_{i=1}^{2} H_{i,t}(y_{i,t}) \right. \\
& \left. + \alpha\beta_O(\mathbf{c}^T\mathbf{y}_t + (\mathbf{p}-\mathbf{c})^T\mathbb{E}[\mathbf{D}_t]) + \sum_{i=1}^{2} \mathbb{E}\Big[\widetilde{V}_{i,t+1}(y_{i,t} - D_{i,t})\Big] \right\}, \\
= & \beta_O w_t + \sum_{i=1}^{2} \widetilde{V}_{i,t}(x_{i,t}),
\end{aligned} \tag{A7}$$

which completes the induction.

It is well known that in the stationary setting, the optimal inventory policy for problem (A6) is a myopic base-stock policy with reorder points S_i defined by

$$S_i = \arg\min \left\{ (\eta - \beta_O + \alpha\beta_O)c_i y_t + H_{i,t}(y_t) \right\}.$$

Therefore, the optimal policy of the problem in (A4) is as follows: the inventory policy is a base-stock policy with base-stock level S_i, while the corporate treasury transfers all excess cash to the investment account after inventory payment.

Finally, we implement the optimal policy of the problem in (A4) to the original problem and show that it can achieve the same expected costs. As a result, the optimal policy of the problem in (A4) is also optimal for the original problem, as the new problem is a lower bound to the original system.

Note that the original and new systems share the same inventory-related costs as they charge identical inventory costs and adopt the same inventory policy. The cost difference arises from the cash-related costs. The problem (A4) charges $\beta_O(w_t - r_t)$, whereas the original problem charges $\beta_I(r_t - w_t)^+ + \beta_O(r_t - w_t)^-$. However, the two cost schemes are equivalent if we can show $w_t \geq r_t$ for every period t, namely, the initial working capital of each period is sufficient for the inventory replenishment. We prove it by the sample path approach. Recall that $r_t = c_1 S_1 + c_2 S_2$. By the assumption that $w_1 \geq c_1 S_1 + c_2 S_2$, the result is true for period 1. Consider a particular demand path $\{(d_{1,1}, d_{2,1}), (d_{1,2}, d_{2,2}), ..., (d_{1,T}, d_{2,T}))\}$. For period 2, the initial working capital is $w_2 = \mathbf{c}^T \mathbf{y}_t + (\mathbf{p}-\mathbf{c})^T \mathbf{d}_t = c_1 S_1 + c_2 S_2 + (\mathbf{p}-\mathbf{c})^T \mathbf{d}_t \geq c_1 S_1 + c_2 S_2 = r_t$. Similarly, one can show that the result still holds for other subsequent periods. □

Proof of Theorem 2. We first prove $V_t(w_t, \mathbf{x}_t) \geq \underline{V}_t(w_t, \mathbf{x}_t|\mathbf{a}_t)$ by induction. As $V_{T+1}(w_{T+1}, \mathbf{x}_{T+1}) = \underline{V}_{T+1}(w_{T+1}, \mathbf{x}_{T+1}|\mathbf{a}_{T+1}) = 0$, the result trivially holds for period $T+1$. Assume that the result is true for period $t+1$, i.e., $V_{t+1}(w_{t+1}, \mathbf{x}_{t+1}) \geq \underline{V}_{t+1}(w_{t+1}, \mathbf{x}_{t+1}|\mathbf{a}_{t+1})$. As $\mathbf{a}_t$ is nonnegative, $\underline{G}_t(w_t, r_t, \mathbf{y}_t|\mathbf{a}_t) \leq G_t(w_t, r_t, \mathbf{y}_t)$. Hence, the objective function in (22) is smaller than that in (12). It immediately follows that $V_t(w_t, \mathbf{x}_t) \geq \underline{V}_t(w_t, \mathbf{x}_t|\mathbf{a}_t)$, as the two optimization problems have the same constraints. The proof of the result $\underline{V}_t(w_t, \mathbf{x}_t|\mathbf{a}_t) \geq \underline{V}_{H,t}(w_t|\mathbf{a}_t) + \sum_{i=1}^{2} \underline{V}_{i,t}(x_{i,t}|\mathbf{a}_t)$ corresponds to the cost allocation scheme stated by Chen and Zheng (1994) and is similar with the proof in Appendix B of their paper. Hence, we omit the proof here. □

Proof of Theorem 3. (i) Division i faces the following optimization problem

$$\underline{V}_{i,t}(x_{i,t}|\mathbf{a}_t) = \min_{y_t \geq x_{i,t}} \Big\{ \underline{J}_{i,t}(y_t|\mathbf{a}_t) - \underline{\Gamma}_{i,t}(y_t|\mathbf{a}_t) \Big\},$$

where $\underline{J}_{i,t}(y_t|\mathbf{a}_t)$ is defined in (23). Note that the cost objective function is

$$\begin{cases} \underline{J}_{i,t}(S_{i,t}(\mathbf{a}_t)|\mathbf{a}_t), & \text{if } \ y_{i,t} \leq S_{i,t}(\mathbf{a}_t); \\ \underline{J}_{i,t}(y_t|\mathbf{a}_t), & \text{otherwise}, \end{cases}$$

where $S_{i,t}(\mathbf{a}_t)$ is defined in (24). It can be readily proven by induction that the problem is convex and a base-stock policy with the base-stock level $S_{i,t}(\mathbf{a}_t)$ is optimal.

We next prove the monotonicity of the base-stock level $S_{i,t}(\mathbf{a}_t)$ with respect to a_τ for any $\tau \geq t$. By the definition of $\underline{\Gamma}_{i,t}(y_t|\mathbf{a}_t)$ in (27), $\underline{V}_{i,t}(x_{i,t}|\mathbf{a}_t)$ can be rewritten as

$$\underline{V}_{i,t}(x_{i,t}|\mathbf{a}_t) = \min_{y_t \geq x_{i,t}} \Big\{ \underline{J}_{i,t}(y_t|\mathbf{a}_t) \Big\}.$$

So we just need to prove $\underline{J}_{i,t}(y_t|\mathbf{a}_t)$ is supermodular in (y_t, a_τ) for all $\tau \geq t$. The result trivially holds for period $T+1$. Assume that $\underline{V}_{i,t+1}(x_{i,t+1}|\mathbf{a}_{t+1})$ is supermodular in $(x_{i,t+1}, a_\tau)$ for $\tau \geq t+1$. Because supermodularity can be preserved under addition and positive scalar multiplication, it follows that the expectation $\alpha \mathbb{E}[\underline{V}_{i,t+1}(y_t - D_{i,t}|\mathbf{a}_{t+1})]$ is supermodular in (y_t, a_τ) for $\tau \geq t+1$. One also can easily verify that the term $H_{i,t}(y_t) + a_t c_i y_t$ is supermodular in (y_t, a_t). Therefore, $\underline{J}_{i,t}(y_t|\mathbf{a}_t)$ is supermodular in (y_t, a_τ) for $\tau \geq t$. To complete the induction, we next prove that $\underline{V}_{i,t}(x_{i,t}|\mathbf{a}_t)$ is supermodular in $(x_{i,t}, a_\tau)$ for all $\tau \geq t$.

We can derive the partial derivative of $\underline{V}_{i,t}(x_{i,t}|\mathbf{a}_t)$ with respect to $x_{i,t}$

$$\frac{\partial \underline{V}_{i,t}(x_{i,t}|\mathbf{a}_t)}{\partial x_{i,t}} = \begin{cases} 0, & \text{if } \ x_{i,t} \leq S_{i,t}(\mathbf{a}_t); \\ \dfrac{\partial \underline{J}_{i,t}(x_{i,t}|\mathbf{a}_t)}{\partial x_{i,t}}, & \text{otherwise}. \end{cases} \tag{A8}$$

To prove the supermodularity, we need to verify that $\frac{\partial \underline{V}_{i,t}(x_{i,t}|\mathbf{a}_t)}{\partial x_{i,t}}$ is increasing in a_τ for $\tau \geq t$. Given a period $\tau \geq t$, let $\mathbf{a}_t^{'} = (a_t, a_{t+1}, ..., a_\tau^{'}, ..., a_T)$ and $\mathbf{a}_t^{''} = (a_t, a_{t+1}, ..., a_\tau^{''}, ..., a_T)$ such that $0 \leq a_\tau^{'} \leq a_\tau^{''}$. As $\underline{J}_{i,t}(y_t|\mathbf{a}_t)$ is supermodular in (y_t, a_τ), the base-stock level $S_{i,t}(\mathbf{a}_t)$ is decreasing in a_τ, i.e., $S_{i,t}(\mathbf{a}_t^{'}) \geq S_{i,t}(\mathbf{a}_t^{''})$. We consider the following three cases.

Case 1. If $x_{i,t} \leq S_{i,t}(\mathbf{a}_t^{''})$, then it follows from (A8) that $\frac{\partial \underline{V}_{i,t}(x_{i,t}|\mathbf{a}_t)}{\partial x_{i,t}}|_{\mathbf{a}_t = \mathbf{a}_t^{'}} = \frac{\partial \underline{V}_{i,t}(x_{i,t}|\mathbf{a}_t)}{\partial x_{i,t}}|_{\mathbf{a}_t = \mathbf{a}_t^{''}} = 0$.

Case 2. If $S_{i,t}(\mathbf{a}_t^{''}) < x_{i,t} \le S_{i,t}(\mathbf{a}_t^{'})$, then $\frac{\partial \underline{V}_{i,t}(x_{i,t}|\mathbf{a}_t)}{\partial x_{i,t}}|_{\mathbf{a}_t=\mathbf{a}_t^{'}} = 0$ and $\frac{\partial \underline{V}_{i,t}(x_{i,t}|\mathbf{a}_t)}{\partial x_{i,t}}|_{\mathbf{a}_t=\mathbf{a}_t^{''}} = \frac{\partial \underline{J}_{i,t}(x_{i,t}|\mathbf{a}_t)}{\partial x_{i,t}}|_{\mathbf{a}_t=\mathbf{a}_t^{''}}$. The convexity of $\underline{J}_{i,t}(\cdot|\mathbf{a}_t)$ implies $\frac{\partial \underline{J}_{i,t}(x_{i,t}|\mathbf{a}_t)}{\partial x_{i,t}}|_{\mathbf{a}_t=\mathbf{a}_t^{''}} \ge 0$. Therefore $\frac{\partial \underline{V}_{i,t}(x_{i,t}|\mathbf{a}_t)}{\partial x_{i,t}}|_{\mathbf{a}_t=\mathbf{a}_t^{'}} \le \frac{\partial \underline{V}_{i,t}(x_{i,t}|\mathbf{a}_t)}{\partial x_{i,t}}|_{\mathbf{a}_t=\mathbf{a}_t^{''}}$.

Case 3. If $S_{i,t}(\mathbf{a}_t^{'}) < x_{i,t}$, $\frac{\partial \underline{V}_{i,t}(x_{i,t}|\mathbf{a}_t)}{\partial x_{i,t}}|_{\mathbf{a}_t=\mathbf{a}_t^{'}} = \frac{\partial \underline{J}_{i,t}(x_{i,t}|\mathbf{a}_t)}{\partial x_{i,t}}|_{\mathbf{a}_t=\mathbf{a}_t^{'}}$ and $\frac{\partial \underline{V}_{i,t}(x_{i,t}|\mathbf{a}_t)}{\partial x_{i,t}}|_{\mathbf{a}_t=\mathbf{a}_t^{''}} = \frac{\partial \underline{J}_{i,t}(x_{i,t}|\mathbf{a}_t)}{\partial x_{i,t}}|_{\mathbf{a}_t=\mathbf{a}_t^{''}}$. The supermodularity of $\underline{J}_{i,t}(\cdot|\mathbf{a}_t)$ implies $\frac{\partial \underline{J}_{i,t}(x_{i,t}|\mathbf{a}_t)}{\partial x_{i,t}}|_{\mathbf{a}_t=\mathbf{a}_t^{'}} \le \frac{\partial \underline{J}_{i,t}(x_{i,t}|\mathbf{a}_t)}{\partial x_{i,t}}|_{\mathbf{a}_t=\mathbf{a}_t^{''}}$. Therefore $\frac{\partial \underline{V}_{i,t}(x_{i,t}|\mathbf{a}_t)}{\partial x_{i,t}}|_{\mathbf{a}_t=\mathbf{a}_t^{'}} \le \frac{\partial \underline{V}_{i,t}(x_{i,t}|\mathbf{a}_t)}{\partial x_{i,t}}|_{\mathbf{a}_t=\mathbf{a}_t^{''}}$.

Therefore, $\frac{\partial \underline{V}_{i,t}(x_{i,t}|\mathbf{a}_t)}{\partial x_{i,t}}$ is increasing in a_τ, and the result holds for any $\tau \ge t$. Hence $\underline{V}_{i,t}(x_{i,t}|\mathbf{a}_t)$ is supermodular in $(x_{i,t}, a_\tau)$ for $\tau \ge t$. The induction completes. As a global minimizer of a supermodular function, the base-stock level $S_{i,t}(\mathbf{a}_t)$ is nonincreasing in the value of a_τ for $\tau \ge t$ given any period t.

(ii) The cash subsystem H solves the following optimization problem

$$\underline{V}_{H,t}(w_t|\mathbf{a}_t) = \min_{r_t \ge \mathbf{c}^T\mathbf{y}_t} \Big\{(\eta - a_t)r_t + F(r_t - w_t) + \sum_{i=1}^{2} \underline{\Gamma}_{i,t}(y_{i,t}|\mathbf{a}_t) + \alpha\mathbb{E}[\underline{V}_{H,t+1}(r_t + (\mathbf{p}-\mathbf{c})^T\mathbf{D}_t|\mathbf{a}_{t+1})]\Big\}.$$

It is easy to verify that this dynamic optimization is a convex problem, and the optimal cash retention policy is a two-threshold policy with the upper and lower thresholds defined in (29) and (30). We refer the reader to Luo and Shang (2015) for the detailed proof of the policy structure. □

Proof of Theorem 4. We first show by induction that the Lagrange relaxation problem defined in (32) is a lower bound to our proposed bound as long as $\boldsymbol{\lambda}_1 = \mathbf{a}_1$. The result trivially holds for period $T+1$. We assume it holds in period $t+1$, i.e., $V_{t+1}^L(w_{t+1}, \mathbf{x}_{t+1}|\boldsymbol{\lambda}_{t+1}) \le \underline{V}_{H,t+1}(w_{t+1}|\boldsymbol{a}_{t+1}) + \sum_{i=1}^2 \underline{V}_{i,t+1}(x_{i,t+1}|\boldsymbol{a}_{t+1})$ when $\boldsymbol{\lambda}_{t+1} = \mathbf{a}_{t+1}$. As a result, we have

$$\begin{aligned}
V_t^L(w_t, \mathbf{x}_t|\boldsymbol{\lambda}_t) &= \min_{\substack{r_t\in\mathbb{R}\\ \mathbf{y}_t \ge \mathbf{x}_t}} \Big\{\eta r_t + F(r_t - w_t) + \sum_{i=1}^2 H_{i,t}(y_{i,t}) - \lambda_t(r_t - \mathbf{c}^T\mathbf{y}_t) + \alpha\mathbb{E}\big[V_{t+1}^L(r_t + (\mathbf{p}-\mathbf{c})^T\mathbf{D}_t, \mathbf{y}_t - \mathbf{D}_t|\boldsymbol{\lambda}_{t+1})\big]\Big\} \\
&\le \min_{\substack{r_t\in\mathbb{R}\\ \mathbf{y}_t \ge \mathbf{x}_t}} \Big\{\eta r_t + F(r_t - w_t) + \sum_{i=1}^2 H_{i,t}(y_{i,t}) - a_t(r_t - \mathbf{c}^T\mathbf{y}_t) + \alpha\mathbb{E}\big[\underline{V}_{H,t+1}(r_t + (\mathbf{p}-\mathbf{c})^T\mathbf{D}_t|\boldsymbol{a}_{t+1}) \\
&\qquad + \sum_{i=1}^2 \underline{V}_{i,t+1}(y_{i,t} - D_{i,t}|\boldsymbol{a}_{t+1})\big]\Big\} \\
&= \min_{r_t\in\mathbb{R}} \Big\{(\eta - a_t)r_t + F(r_t - w_t) + \alpha\mathbb{E}[\underline{V}_{H,t+1}(r_t + (\mathbf{p}-\mathbf{c})^T\mathbf{D}_t|\mathbf{a}_{t+1})]\Big\} \\
&\qquad + \sum_{i=1}^2 \Big[\min_{y_{i,t}\ge x_{i,t}} \Big\{H_{i,t}(y_{i,t}) + a_t c_i y_{i,t} + \alpha\mathbb{E}[\underline{V}_{i,t+1}(y_{i,t} - D_{i,t}|\mathbf{a}_{t+1})]\Big\}\Big], \\
&\le \min_{r_t\in\mathbb{R}} \Big\{(\eta - a_t)r_t + F(r_t - w_t) + \alpha\mathbb{E}[\underline{V}_{H,t+1}(r_t + (\mathbf{p}-\mathbf{c})^T\mathbf{D}_t|\mathbf{a}_{t+1})] + \min_{\mathbf{c}^T\mathbf{y}_t \le r_t} \sum_{i=1}^2 \underline{\Gamma}_{i,t}(y_{i,t}|\mathbf{a}_t)\Big\} \\
&\qquad + \sum_{i=1}^2 \Big[\min_{y_{i,t}\ge x_{i,t}} \Big\{H_{i,t}(y_{i,t}) + a_t c_i y_{i,t} + \alpha\mathbb{E}[\underline{V}_{i,t+1}(y_{i,t} - D_{i,t}|\mathbf{a}_{t+1})]\Big\}\Big], \\
&= \underline{V}_{H,t}(w_t|\boldsymbol{a}_t) + \sum_{i=1}^2 \Big[\min_{y_{i,t}\ge x_{i,t}} \Big\{H_{i,t}(y_{i,t}) + a_t c_i y_{i,t} - \underline{\Gamma}_{i,t}(y_{i,t}|\mathbf{a}_t) + \alpha\mathbb{E}[\underline{V}_{i,t+1}(y_{i,t} - D_{i,t}|\mathbf{a}_{t+1})]\Big\}\Big], \\
&= \underline{V}_{H,t}(w_t|\boldsymbol{a}_t) + \sum_{i=1}^2 \underline{V}_{i,t}(x_{i,t}|\boldsymbol{a}_t), \qquad \text{(A9)}
\end{aligned}$$

where the first inequality follows from the induction assumption and the second from the non-negativity of $\underline{\Gamma}_{i,t}(y_{i,t}|\mathbf{a}_t)$ which are defined in (27). That is, the result is also true for period t when $\boldsymbol{\lambda}_t = \mathbf{a}_t$ and hence the induction completes.

As the simplified problem is convex and satisfies the Slater's condition, there is no duality gap, i.e., $V_t^L(w_t, \mathbf{x}_t|\boldsymbol{\lambda}_t^*) = V_t(w_t, \mathbf{x}_t)$ where $\boldsymbol{\lambda}_t^*$ is the dual optimal multipliers. By (A9), $\underline{V}_{H,t}(w_t|\boldsymbol{\lambda}_t^*) + \sum_{i=1}^2 \underline{V}_{i,t}(x_{i,t}|\boldsymbol{\lambda}_t^*) \geq V_t^L(w_t, \mathbf{x}_t|\boldsymbol{\lambda}_t^*) = V_t(w_t, \mathbf{x}_t)$. However, Theorem 2 implies that $V_t(w_t, \mathbf{x}_t) \geq \underline{V}_{H,t}(w_t|\boldsymbol{\lambda}_t^*) + \sum_{i=1}^2 \underline{V}_{i,t}(x_{i,t}|\boldsymbol{\lambda}_t^*)$. Therefore, we have $V_t^L(w_t, \mathbf{x}_t|\boldsymbol{\lambda}_t^*) = \underline{V}_{H,t}(w_t|\boldsymbol{\lambda}_t^*) + \sum_{i=1}^2 \underline{V}_{i,t}(x_{i,t}|\boldsymbol{\lambda}_t^*) = V_t(w_t, \mathbf{x}_t)$. □

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