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Serial Supply Chains with Economies of Scale: Bounds and Approximations

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We consider two models of stochastic serial inventory systems with economies of scale for which the forms of optimal policies are known. In the first model, each stage has a fixed-order quantity, while in the second model, there is a fixed-order cost for external supplies. For each model, we show that the optimal policy parameters can be bounded and approximated by a series of independent, single-stage optimal policy parameters. We further construct closed-form bounds and approximations for the single-stage solutions and apply them to the serial systems. These results provide simple and effective solutions that will help to facilitate implementations in practice. They also allow us to see the connections between the serial and single-stage systems and sharpen our intuition on optimal policy parameters and system behavior.

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1. Introduction

Economy of scale is one of the primary concerns when companies design and manage their supply chain networks. To ensure timely delivery of the products at the lowest possible cost, they must pay close attention to the coordination of the shipments from one stage to the next throughout the supply chain. This explains why many researchers have studied multiechelon inventory systems with economies of scale; see, for example, Clark and Scarf (1960, 1962), de Bolt and Graves (1985), Axsäter (1993), Axsäter and Rosling (1993), Chen and Zheng (1994a, 1998), Chen (1998, 2000), Cachon (2001), Doğru et al. (2005), and the references therein. See also the reviews by Federgruen (1993), Chen (1999), and Axsäter (2003). Although tremendous progress has been made in identifying effective control policies and developing efficient algorithms for performance evaluation and optimization, our qualitative understanding of such systems remains limited. For instance, while we can compute optimal policy parameters by feeding problem data into computer programs, it does not help us to see how optimal solutions depend on system parameters. The purpose of this paper is to increase the transparency of such systems by linking a multistage, serial system with the much better understood single-stage systems. Our effort is in line with and also built on previous works on serial systems without economies of scale; see, for example, Gallego and Zipkin (1999), Shang and Song (2003), Dong and Lee (2003), and Gallego and Özer (2005). In particular, our analysis extends that of Shang and Song (2003).

We consider continuous-review serial inventory systems with the objective of minimizing long-run average system-wide costs. There is a total of N stages. Random demand occurs at stage 1 and follows a Poisson process with rate λ. (The extension to the compound Poisson demand case is discussed in §5.) Stage 1 replenishes its stock from stage 2, which, in turn, obtains replenishment from stage 3, and so on. Finally, stage N orders from an outside ample supplier. There is constant transportation lead time L_j between stage $j+1$ and stage j, $j<N$. There is a linear echelon holding cost with rate h_j at stage j. Denote $h[i,j]=\sum_{k=i}^{j} h_k$, and the local holding-cost rate $h'_j = h[j,N]$. Unsatisfied demand is fully backlogged and incurs a linear penalty cost with rate b. A general model to account for economies of scale consists of a fixed and a variable ordering cost at each stage. Unfortunately, the form of the optimal policy for this general model is extremely difficult to characterize, and even if it is known, it would be very complex; see Clark and Scarf (1962). Nonetheless, there are two important special cases for which the forms of the optimal policies have been identified, and we shall focus on them in this paper.

The first model assumes that there is a fixed-order quantity q_j at each stage j, such as a truckload or the size of a standard container. Also, the order quantities of the stages satisfy an integer-ratio constraint, i.e., $q_{j+1}=n_j q_j$, where n_j is a positive integer. Under this assumption, Chen (2000) shows that an echelon (r,q) policy is optimal and presents an algorithm to compute optimal reorder points. Recently, Doğru et al. (2005) derived an alternative algorithm based on the concept of shortfall. We denote this system by $R(N,(q_i,h_i,D_i)_{i=1}^{N},b)$, where R indicates that the reorder points are the decision variables.

The second model assumes that there is a fixed-order cost only for external orders, i.e., $k_N > 0$ and $k_j = 0$, $j < N$. The optimal policy is an echelon (r, q) policy at stage N and an echelon base-stock policy for each of the downstream stages (Clark and Scarf 1960, Federgruen and Zipkin 1984, and Chen and Zheng 1994b). The decision variables are the reorder point r_N, order quantity q_N, and the base-stock levels s_j, $j < N$. A recursive optimization algorithm can be found in Chen and Zheng (1994b). We denote this system by $RQ(N, k_N, (h_i, D_i)_{i=1}^N, b)$, where RQ indicates that both the reorder points at all stages and the order quantity at the upmost stage are the decision variables.

For each model, we show that the optimal policy parameters can be bounded and approximated by a series of independent, single-stage optimal (r, q) policy parameters. To further identify key determinants of optimal policies, we devise closed-form bounds and approximations for optimal solutions and costs of single-stage (r, q) systems, building on early works by Zheng (1992) and Gallego (1998), and then apply these results to the serial system. Numerical experiments show that these closed-form expressions are close approximates of the optimal solutions. We then perform sensitivity analysis based on these closed-form formulas.

The rest of this paper is organized as follows. Section 2 reviews and develops closed-form bounds on the optimal reorder point and cost for the single-stage system. Section 3 focuses on the first model. Both single-stage and closed-form bounds and approximation for the optimal reorder point are presented. Qualitative insights from parametric analysis are also summarized. Section 4 concentrates on the second model. Section 5 extends the results to the compound Poisson demand case. Section 6 summarizes the main findings. All proofs are in the appendix.

2. Single-Stage Systems

In this section, we develop closed-form bounds on the optimal reorder points and costs for single-stage systems. These results will help us establish linkages of optimal policies among different stages in the serial systems in later sections. Because $N = 1$, we shall suppress the subscript j in the notation.

Let $x^+ = \max\{0, x\}$, $x^- = \max\{0, -x\}$. The average cost of the single-stage (r, q) system can be expressed as $C(r, q) = (k\lambda + \sum_{x=1}^{q} G(r + x))/q$. Here,

$$G(y) = \mathsf{E}[h(y - D)^+ + b(y - D)^-] \tag{1}$$

is the average cost of a base-stock system with base-stock level y, which is also known as the newsvendor cost function. Denote by $r^*(q)$ the optimal reorder point that minimizes $C(r, q)$ for any fixed q. Let q^* be the optimal solution that minimizes $C(r(q), q)$ over q and let $r^* = r^*(q^*)$. Then, $r^*(q)$ is the optimal policy for $R(1, q, h, D, b)$, and (r^*, q^*) is the optimal policy for $RQ(1, k, h, D, b)$; see, for example, Zipkin (2000). (In the following, for simplicity we use r^* to indicate the optimal reorder point in either system.)

The basic idea of establishing the bounds on r^* is to relate the (r, q) model with two simpler models. The first one is the base-stock system that has no economies of scale, whose cost is given by $G(y)$. The second model is the economic order quantity (EOQ) model that ignores demand uncertainty, whose cost is $C(r, q)$ with $\mathsf{E}[D]$ replacing D in (1). For convenience, we shall approximate the Poisson variable D by a normal distribution. This approximation is appropriate when the mean lead time demand $\mathsf{E}[D]$ is sufficiently large. (When $\mathsf{E}[D]$ is small, one may use the maximal approximation; see Zipkin 2000, p. 216. Also, see the remark in §5.) Correspondingly, we shall treat r and q as continuous variables.

Let $\Phi(\cdot)$ and $\phi(\cdot)$ be the c.d.f. and the p.d.f. of the standard normal distribution. Define $\omega = b/(b + h)$. Then, the optimal solution and cost of the base-stock system (1) are $s^* = \mathsf{E}[D] + z^*\sqrt{\mathsf{Var}[D]}$ and $c_s = (b + h)\phi(z^*)\sqrt{\mathsf{Var}[D]}$, respectively, where $z^* = \Phi^{-1}(\omega)$ is the safety factor. For the EOQ model, the optimal order quantity and cost are given by $q_d = \sqrt{2k\lambda/(h\omega)}$ and $c_d = \sqrt{2k\lambda h\omega}$. We have:

Proposition 1. *The optimal reorder point satisfies* $r_- \leqslant r^* \leqslant r_+$, *where* (1) *in the* $R(1, q, h, D, b)$ *system,* $r_- = \mathsf{E}[D] - (1 - \omega)q - c_s/b$, $r_+ = s^* - (1 - \omega)q$, *and* (2) *in the* $RQ(1, k, h, D, b)$ *system,* $r_- = \mathsf{E}[D] - (1 - \omega)q_d - c_s/b$, $r_+ = s^* - (1 - \omega)q_d$.

Note that when $b > h$, $r^* \geqslant s^* - 0.5q$ (Zipkin 2000, p. 218). Thus, we can obtain an improved lower-bound $r_- = \max\{s^* - 0.5q, \mathsf{E}[D] - (1 - \omega)q - c_s/b\}$ for r^* in Proposition 1(1). It can be shown that when $q/\sqrt{\mathsf{Var}[D]} \geqslant (z^* - \phi(z^*)/\omega)/(\omega - 0.5)$, the second component in r_- is active. For example, when $\omega = 0.9$, this condition implies $q \geqslant 2.716\sqrt{\mathsf{Var}[D]}$.

We next construct closed-form bounds for the optimal cost. Let $c_{s+} = \sqrt{bh\mathsf{Var}[D]}$.

Proposition 2. *The optimal cost satisfies the following*: (1) *in the* $R(1, q, h, D, b)$ *system,* $\max\{c_s, qh\omega\} \leqslant C(r^*, q) \leqslant qh\omega + c_s$, *and* (2) *in the* $RQ(1, k, h, D, b)$ *system,* $\sqrt{c_d^2 + c_s^2} \leqslant C(r^*, q^*) \leqslant \sqrt{c_d^2 + c_{s+}^2}$ (*Gallego* 1998).

3. Serial Systems with Fixed Base-Order Quantities

In this section, we focus on the serial system $R(N, (q_i, h_i, D_i)_{i=1}^N, b)$. We first review the existing results for policy evaluation and optimization, drawn from Chen (2000). The dynamics of this system in steady state can be characterized by IP_j and IN_j, the echelon inventory position and echelon net inventory level at stage j, respectively, for all j. In particular, IP_N is uniformly distributed over

$\{r_N+1, \ldots, r_N+q_N\}$, and $IN_j = IP_j - D_j$, $j = 1, \ldots, N$. Also, $IP_j = O_j[IN_{j+1}]$, $j = 1, \ldots, N-1$, where

$$O_j[x] = \begin{cases} x & \text{if } x \leqslant r_j, \\ x - mq_j & \text{otherwise,} \end{cases} \tag{2}$$

with m being the largest integer so that $x - mq_j > r_j$. Denote by I'_j the local on-hand inventory at stage j and B the number of backorders at stage 1. We have $I'_j = IN_j - IP_{j-1}$, $j \geqslant 2$, $I'_1 = [IN_1]^+$, and $B = [IN_1]^-$. The long-run average systemwide cost is

$$C(r_1, \ldots, r_N) = \mathsf{E}\left[\sum_{i=1}^{N} h'_i I'_i + bB + \sum_{i=2}^{N} h'_i D_{i-1}\right]$$

$$= \mathsf{E}\left[\sum_{i=1}^{N} h_i IN_i + (b + h'_1)B\right]. \tag{3}$$

The optimal reorder points $(r_1^*, \ldots, r_N^*)$ that minimize the above cost can be obtained recursively as follows. Define C_j as the average inventory holding and backorder costs for echelon-j. For $j = 1, \ldots, N$, let

$$C_j(y) = \begin{cases} \mathsf{E}[h_1(y - D_1) + (b + h'_1)(y - D_1)^-], & j = 1, \\ \mathsf{E}[h_j(y - D_j) + C_{j-1}(O_{j-1}[y - D_j])], \\ \quad j = 2, \ldots, N, \end{cases} \tag{4}$$

$$\bar{C}_j(\cdot) = \sum_{x=1}^{q_j} C_j(\cdot + x) \quad \text{and} \quad r_j^* = \arg\min_y \bar{C}_j(y). \tag{5}$$

Note that in the above equation, O_{j-1} is with respect to r_{j-1}^*. The optimal cost is $C_N^* = C(r_1^*, \ldots, r_N^*) = [\sum_{x=1}^{q_N} C_N \cdot (r_N^* + x)]/q_N$.

While the above procedure computes the exact optimal solution, it is difficult to see the effect of order quantities q_j and other parameters on the optimal solution and cost. To overcome this difficulty, in §3.1 we construct a series independent single-stage (r, q) system to bound the original serial system. This is accomplished in three steps:

Step 1. Re-express the echelon-j cost function $C_j(y)$ in terms of local inventory levels at every stage within the echelon (Proposition 3).

Step 2. Replace the "local" holding-cost rates at all stages in this new expression by a single value to collapse echelon-j into a single-stage (r, q) system. When the single value equals the largest (smallest) holding-cost rate replaced, we obtain an upper (lower) bound system, i.e., the cost functions of these two single-stage (r, q) systems bound the echelon-j cost function (Theorem 4(1)–(3)).

Step 3. Show that the differences of the bounding-cost functions also bound the difference of the echelon-j function (Theorem 4(4)), which, in turn, implies that the optimal reorder points obtained from the two single-stage bounding systems bound the optimal echelon reorder point r_j^* (Theorem 4(5)).

These steps resemble those used in Shang and Song (2003) for base-stock systems. Based on the results in §§2 and 3.1, in §§3.2 and 3.3 we develop single-stage and closed-form approximations to the optimal policy parameters. We then report numerical results on the effectiveness of these approximations in §3.4 and conduct sensitivity analysis in §3.5.

3.1. Single-Stage Bounds

From the recursions (4) and (5), we observe that for each stage j, the optimal reorder point r_j^* does not depend on the decisions at upstream stages. To determine r_j^*, the echelon-j manager only needs to know b, h'_1, and (r_i^*, q_i, D_i, h_i), $i < j$.

More specifically, conditional on $IP_j = y$, let $I'_i(y)$ denote the local on-hand inventory at stage i, $i \leqslant j$, and $B(y)$ the number of backorders at stage 1 in system-j, assuming the echelon (r_i^*, q_i) policy is employed at stage i, $i < j$. Denote $\tilde{D}_j = \sum_{i=1}^{j} D_j$. Then, we can obtain the following decomposition for $C_j(y)$. The proof is omitted.

Proposition 3. *For each $j \geqslant 2$ and conditioning on $IP_j = y$, $C_j(y)$ is the average inventory holding and backorder costs for system-j assuming that the echelon (r_i^*, q_i) policy at stage i, $i < j$ is employed, and $C_j(y) = G_j(y) + \tau_j$, where*

$$G_j(y) = \mathsf{E}[h_j I'_j(y) + h[j, j-1] I'_{j-1}(y) + \cdots + h[1, j] I'_1(y) + (b + h'_{j+1}) B(y)], \text{ and}$$

$$\tau_j = \text{the average in-transit holding cost in system-}j \tag{6}$$

$$= \sum_{i=2}^{j} (h[i, j]) \mathsf{E}[D_{i-1}] = \sum_{i=2}^{j} h_i \mathsf{E}[\tilde{D}_{i-1}].$$

Comparing with (3), Proposition 3 implies that, under the echelon policy (r_i^*, q_i) for $i < j$, the echelon-j manager in effect faces a system with the local holding-cost rate $h[i, j]$ for stage i, $i \leqslant j$, and backorder cost rate $(b + h'_{j+1})$ at stage 1. In other words, the system has exactly the same structure as $R(j, (q_i, h_i, D_i)_{i=1}^{j}, b + h'_{j+1})$, a truncated j-stage system consisting of the stages $1, 2, \ldots, j$ of the original system, but now stage j has ample supply. For convenience, we term this system as system-j.

Now, for any y, according to (6), if we replace the different holding-cost rates at different stages in system-j by a single common value, and if we relax the batch shipment constraint, then there would be no incentive to carry inventories at stage i, $2 \leqslant i \leqslant j$, and the j-stage system would collapse into a single-stage base-stock system with the total lead time $\tilde{L}_j = \sum_{i=1}^{j} L_i$ and base-stock level y.

By setting this single common value to the minimum holding-cost rate h_j and relaxing the constraint of batch shipping, we then obtain a single-stage lower-bound base-stock system, whose cost function is

$$G_j^l(y) = \mathsf{E}[h_j(y - \tilde{D}_j)^+ + (b + h'_{j+1})(y - \tilde{D}_j)^-],$$
$$\underline{\tau}_j = h_j \mathsf{E}[\tilde{D}_{j-1}]. \tag{7}$$

Similarly, by setting the single holding-cost-rate value to the maximum holding-cost rate $\sum_{i=1}^{j} h_i$ in system-j and eliminating the batch shipment requirement, we obtain a single-stage upper-bound base-stock system, whose cost function is

$$G_j^u(y) = \mathsf{E}[h[1,j](y-\widetilde{D}_j)^+ + (b+h'_{j+1})(y-\widetilde{D}_j)^-],$$
$$\bar{\tau}_j = h[1,j]\mathsf{E}[\widetilde{D}_{j-1}]. \tag{8}$$

In other words, for any given y, $G_j(y)$ is bounded by two single-stage base-stock cost functions as follows:

$$G_j^l(y) + \underline{\tau}_j \leqslant G_j(y) + \tau_j \leqslant G_j^u(y) + \bar{\tau}_j. \tag{9}$$

Recall that stage j follows an (r_j, q_j) policy. With ample supply in system-j, IP_j is uniformly distributed over $\{r_j + 1, \ldots, r_j + q_j\}$. Taking expectations on all terms in (9) over this uniform distribution, we have

$$\frac{1}{q_j}\sum_{x=1}^{q_j} G_j^l(y+x) + \underline{\tau}_j \leqslant \frac{1}{q_j}\sum_{x=1}^{q_j} G_j(y+x) + \tau_j$$
$$\leqslant \frac{1}{q_j}\sum_{x=1}^{q_j} G_j^u(y+x) + \bar{\tau}_j. \tag{10}$$

That is, the average echelon-j cost is bounded by the costs of the lower-bound (r, q) system $R(1, q_j, h_j, \widetilde{D}_j, b + h'_{j+1})$ and the upper-bound (r, q) system $R(1, q_j, h[1, j], \widetilde{D}_j, b + h'_{j+1})$. Note that Shang and Song (2003) show that the single-stage cost functions G_j^l and G_j^u bound the cost of echelon-j in which base-stock policies are used at downstream stages. Here, we show that the same cost functions bound the cost of echelon-j in which (r, q) policies are used at downstream stages.

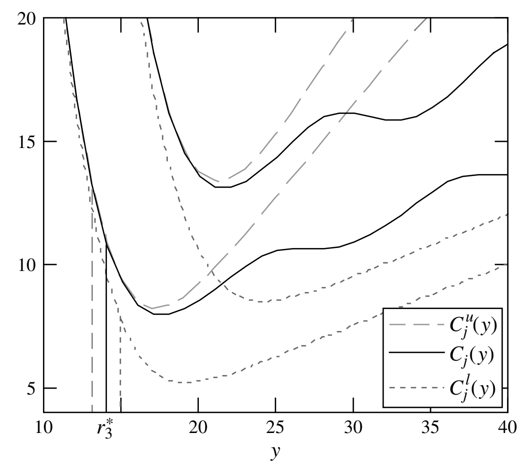
Observe that the average in-transit holding cost τ_j in the original system is independent of the choice of policies. One may wonder whether $\underline{\tau}_j$ and $\bar{\tau}_j$ can be replaced by τ_j in (9) and (10). Because $\underline{\tau}_j \leqslant \tau_j \leqslant \bar{\tau}_j$, this means to have tighter bounds. Theorem 4 below shows that this is indeed true.

Define $C_j^l(y) = G_j^l(y) + \tau_j$, $C_j^u(y) = G_j^u(y) + \tau_j$, $\bar{C}_j^l(y) = \sum_{x=1}^{q_j} C_j^l(y+x)$, $\bar{C}_j^u(y) = \sum_{x=1}^{q_j} C_j^u(y+x)$, $r_j^u = \arg\min_y\{\bar{C}_j^l(y)\}$, and $r_j^l = \arg\min_y\{\bar{C}_j^u(y)\}$. Let $\Delta f(x) = f(x+1) - f(x)$ for any discrete function f. The main results of this section are summarized in the following theorem.

Theorem 4. *For* $j = 1, \ldots, N$, (1) $C_j^l(y) \leqslant C_j(y) \leqslant C_j^u(y)$. (2) $G_j^l(y) \leqslant G_j(y) \leqslant G_j^u(y)$. (3) $\bar{C}_j^l(y) \leqslant \bar{C}_j(y) \leqslant \bar{C}_j^u(y)$. (4) $\Delta\bar{C}_j^l(y) \leqslant \Delta\bar{C}_j(y) \leqslant \Delta\bar{C}_j^u(y)$. (5) $r_j^l \leqslant r_j^* \leqslant r_j^u$. *When* $j = 1$, *all the above inequalities reduce to equalities.*

It is interesting to note that although for each j, $C_j(y)$ is not convex in y, Theorem 4(1) shows that it is bounded by two convex functions $C_j^l(y)$ and $C_j^u(y)$ for all y. Moreover, with the same q_j, the optimal reorder point r_j^* of the nonconvex function is also bounded by r_j^u and r_j^l on the two convex functions. Figure 1 provides an example of a four-stage system. Here, we plot $C_j(y)$, $j = 3, 4$, and their bounding functions. We also plot r_3^* and its bounds.

Figure 1. Illustration of cost functions and bounds for a four-stage system with $h_j = 0.25$, $L_j = 0.25$ $\forall\, j$, $\lambda = 16$, $b = 9$, $q_1 = 3$, $q_2 = 12$, $q_3 = 12$, $q_4 = 24$.



Let the optimal cost for the lower-bound system $R(1, q_N, h_N, \widetilde{D}_N, b)$ be $C_N^l = (1/q_N)\bar{C}_N^l(r_N^u)$ and for the upper-bound system $R(1, q_N, h'_1, \widetilde{D}_N, b)$ be $C_N^u = (1/q_N)\bar{C}_N^u(r_N^l)$. Note that $\bar{C}_N^l(r_N^u) \leqslant \bar{C}_N^l(r_N^*) \leqslant \bar{C}_N(r_N^*) \leqslant \bar{C}_N(r_N^l) \leqslant \bar{C}_N^u(r_N^l)$. Thus, we have:

Proposition 5. $C_N^l \leqslant C_N^* \leqslant C_N^u$.

3.2. Single-Stage Heuristics

In general, there are two ways of constructing approximations for the optimal reorder points.

(1) According to Theorem 4(5), any convex combination of r_j^l and r_j^u can be used to approximate r_j^*.

(2) We can replace the coefficients of $I_i'(y)$ in Proposition 3 by a single convex combination of them to obtain a base-stock system (similar to the bounding systems) and use its solution as the approximate solution.

In principle, extensive numerical experiments can be carried out to identify effective weights. We term the heuristic developed under these approaches as the *single-stage heuristic* (SSH).

3.3. Closed-Form Bounds and Heuristics

We now develop closed-form approximations for the single-stage bounds. Let $\omega_j^l = (b + h'_{j+1})/(b + h'_1)$ and $\omega_j^u = (b + h'_{j+1})/(b + h'_j)$. Denote by $F_j(\cdot)$ the c.d.f. of $\widetilde{D}_j$, and define $F_j^{-1}(\omega) = \min\{y \mid F_j(y) > \omega\}$, $0 \leqslant \omega \leqslant 1$. Then, we can express the minimizers of $G_j^l(\cdot)$ and $G_j^u(\cdot)$ in (7) and (8) as follows:

$$s_j^l = F_j^{-1}(\omega_j^l) \quad \text{and} \quad s_j^u = F_j^{-1}(\omega_j^u). \tag{11}$$

Suppose that lead time demand $\tilde{D}_j$ is sufficiently large, and we approximate $\tilde{D}_j$ by a normal distribution with $\mathsf{E}[\tilde{D}_j]=\lambda\tilde{L}_j$ and $\mathsf{Var}[\tilde{D}_j]=\lambda\tilde{L}_j$. Then, we can apply the results in §2 to construct closed-form bounds and approximation for r_j^*. Specifically, let $s_j^l=\mathsf{E}[\tilde{D}_j]+z_j^l\sqrt{\mathsf{Var}[\tilde{D}_j]}$ and $s_j^u=\mathsf{E}[\tilde{D}_j]+z_j^u\sqrt{\mathsf{Var}[\tilde{D}_j]}$, where $z_j^l=\Phi^{-1}(\omega_j^l)$ and $z_j^u=\Phi^{-1}(\omega_j^u)$. Also, $G_j^u(s_j^l)=(b+h_1')\phi(z_j^l)\sqrt{V[\tilde{D}_j]}$. We have:

Corollary 6. *The optimal echelon reorder points satisfy $r_{j-}^l\leqslant r_j^*\leqslant r_{j+}^u$, $j=1,\ldots,N$, where $r_{j-}^l=s_j^l-0.5q_j$, $\mathsf{E}[\tilde{D}_j]-(1-\omega_j^l)q_j-G_j^u(s_j^l)/(b+h_{j+1}')$, $r_{j+}^u=s_j^u-(1-\omega_j^u)q_j$.*

Note that when $b>h_1'$, we can obtain an improved lower-bound $r_j^l=\max\{s_j^l-0.5q_j,\mathsf{E}[\tilde{D}_j]-(1-\omega_j^l)q_j-G_j^u(s_j^l)/(b+h_{j+1}')\}$.

Closed-Form Heuristics. We can use a convex combination of r_{j-}^l and r_{j+}^u to obtain a closed-form approximation for r_j^*, denoted by r_j^c. We term this heuristic approach as the *closed-form heuristic* (CFH).

3.4. Numerical Study

We now present the results of our numerical study to test the effectiveness of the single-stage bounds r_j^l, r_j^u, heuristic solution r_j^a, and the closed-form counterparts r_{j-}^l, r_{j+}^u, and r_j^c. Among the many possibilities for choosing r_j^a and r_j^c, we chose to report here a special case: the simple average of the upper and lower bounds. That is, $r_j^a=\langle(r_j^l+r_j^u)/2\rangle$ and $r_j^c=\langle(r_{j-}^l+r_{j+}^u)/2\rangle$. Here, $\langle\cdot\rangle$ is the rounding operator, which takes the integer part of the operand. Let C_N^*, C_N^a, and C_N^c denote the costs of the optimal policy, SSH, and CFH, respectively. We define the percentage error of a heuristic policy as

$$\text{error }(\%)=\frac{(\text{heuristic cost}-C_N^*)100\%}{C_N^*}. \qquad (12)$$

We use $(r_j^u-r_j^l)/r_j^*\times 100\%$ and $(r_{j+}^u-r_{j-}^l)/r_j^*\times 100\%$ to measure the relative gaps of the single-stage (SS) bounds and the closed-form (CF) bounds to the optimal reorder point, respectively. Similarly, we use $|r_j^a-r_j^*|/r_j^*\times 100\%$ and $|r_j^c-r_j^*|/r_j^*\times 100\%$ to denote the relative gaps between the heuristic and the optimal solutions, under SSH and CFH, respectively.

We consider linear holding-cost form $h_j=1/N$, affine holding-cost form $h_N=\alpha+(1-\alpha)/N$, $h_j=(1-\alpha)/N$, $j<N$, kink holding-cost form $h_j=(1-\alpha)/N$, $j\leqslant N/2$, $h_j=(1+\alpha)/N$, $j>N/2$, and jump holding-cost form $h_{N/2}=\alpha+(1-\alpha)/N$, $h_j=(1-\alpha)/N$, $j\neq N/2$, in a four-stage system with $\alpha=0.75$. Similarly, we consider these four different forms of lead times by replacing h with L and setting $\alpha=0.25$ in the above formulas. We fixed $q_1=3$ and $q_4=24$, so there are total of 10 different sets of q_2 and q_3 that satisfy the integer-ratio constraints. In all 160 instances, we set $b=39$ and $\lambda=32$. Because $b>h_1'$, we use the improved lower-bound for r_{j-}^l. Table 1 summarizes the result: the first two columns show the average and maximum percentage cost error; the third and the forth columns report the average and maximum spread of solution bounds to the optimal solution for all 640 (=160 × 4) stages; and the last two columns report the average and maximum gap between the heuristic solution and the optimal solution.

Table 1. Summary of the numerical test for Model I.

	Cost		Spread of bounds		Solution	
(%)	Avg.	Max.	Avg.	Max.	Avg.	Max.
SSH	0.09	0.59	5.40	25.00	0.90	6.67
CFH	0.29	1.73	24.55	55.67	2.56	10.53

From these results, we observe that both heuristic policies (SSH and CFH) are fairly effective; their relative cost errors are small. Although the gaps between the solution bounds are large in some cases, the optimal solution tends to be located close to the middle of the solution bounds. Moreover, from the detailed data (not reported here), we observe that CFH demonstrates a similar behavior as the optimal policy when a system parameter changes. This property enables us to predict optimal system behaviors by conducting sensitivity analysis on the heuristic solution.

3.5. Parametric Analysis and Qualitative Insights

As mentioned in the previous subsection, r_j^* is tightly bounded by r_j^l and r_j^u, and the direction of change is the same as r_j^a. We now use these approximations to perform parametric analysis.

First, note that

$$r_j^l \text{ (respectively, } r_j^u)$$
$$=\min\left\{y\,\middle|\,\frac{1}{q_j}\sum_{x=1}^{q_j}\mathsf{P}(\tilde{D}_j\leqslant y+x) > \frac{b+h_{j+1}'}{b+h_1'}\left(\text{respectively, }\frac{b+h_{j+1}'}{b+h_j'}\right)\right\}; \qquad (13)$$

see Gallego (1998). Because for any fixed x, $\mathsf{P}(\tilde{D}_j\leqslant y+x)$ increases in y, so does $\sum_{x=1}^{q_j}\mathsf{P}(\tilde{D}_j\leqslant y+x)$. Thus, as long as q_j is fixed, r_j^l and r_j^u respond to changes of the other system parameters in the same fashion as if $q_j=1$. Thus, the results of parametric analysis in Shang and Song (2003) can be directly applied to reorder points. To summarize, we have:

Proposition 7. (1) *As b or λ increases, r_i^l, r_i^u, and r_i^a increase for all i.*

(2) *As h_j increases, r_i^l increases for $i=1,\ldots,j-1$, but decreases for $i=j,\ldots,N$; r_i^u increases for $i=1,\ldots,j-1$, decreases for $i=j$, and remains unchanged for $i=j+1,\ldots,N$; r_i^a increases for $i=1,\ldots,j-1$, but decreases for $i=j,\ldots,N$.*

(3) *As L_j, $j \geqslant 1$ increases, r_i^l, r_i^u, and r_i^a increase for $i = j, \ldots, N$, but remain the same for $i = 1, \ldots, j-1$.*

(4) *As h_j, b, L_j, or λ increases, both $C_N^l(r_N^u)$ and $C_N^u(r_N^l)$ increase.*

(5) *Reducing lead time at a downstream stage or reducing echelon holding cost at an upstream stage leads to a larger cost reduction.*

Next, we discuss the effect of q_j on the optimal reorder point r_j^* and the optimal cost C_N^* by using Corollary 6 and Proposition 2. Specifically, applying Proposition 2(1) to C_N^*,

$$\max\left\{C_N^l(s_N^u),\, h_N\omega_N^u\left(\prod_{i=1}^{N-1} n_i\right)q_1 + \tau_j\right\} \leqslant C_N^* \leqslant h_1'\omega_N^l\left(\prod_{i=1}^{N-1} n_i\right)q_1 + C_N^u(s_N^l). \tag{14}$$

Thus, we have:

Proposition 8. (1) *r_{j+}^u, r_{j-}^l, and r_j^c decrease in q_j.*

(2) *Both bounds for the optimal cost C_N^* in (14) are linear and increasing in q_1, and concave and increasing in $\tilde{L}_N$.*

Our numerical observations confirm that both r_j^* and C_N^* possess the above properties. Finally, the cost bounds in (14) suggest that by adding one more unit to q_1, the system cost may increase by the magnitude proportional to $\prod_{i=1}^{N-1} n_i$, which can be quite large. We term this phenomenon as the *multiplication effect* on order quantities.

4. Serial Systems with Fixed Costs for External Supplies

This section focuses on the serial system $RQ(N, k_N, (h_i, D_i)_{i=1}^N, b)$. The optimal policy is an echelon (r, q) policy at stage N and an echelon base-stock policy at stage j, $j < N$. The optimal echelon base-stock levels for stages 1 to $N-1$, $(s_1^*, \ldots, s_{N-1}^*)$ can be calculated recursively through the same recursion from (4) to (5) by setting $q_j = 1$ and $s_j^* = r_j^* + 1$ for $j < N$. The optimal policy at stage N, (r_N^*, q_N^*), can be obtained by solving

$$\min_{r_N, q_N} \frac{\lambda k_N + \sum_{x=1}^{q_N} C_N(r_N + x)}{q_N} \tag{15}$$

(Chen and Zheng 1994b). Denote the optimal cost by $C_N^* = C(s_1^*, \ldots, s_{N-1}^*, r_N^*, q_N^*)$.

4.1. Single-Stage Bounds and Heuristic

Steps (1) to (3) in §3.1 are adopted to develop the cost and solution bounds for the optimal base-stock levels at stage j, $j < N$, by setting $q_j = 1$. The resulting bounds are exactly the same as those in Shang and Song (2003). That is, $s_j^l \leqslant s_j^* \leqslant s_j^u$, where s_j^l and s_j^u are defined in (11). Our focus in this section is to construct bounds for stage N.

The construction of bounds is built upon the single-stage cost functions $C_N^l(\cdot)$ and $C_N^u(\cdot)$. With fixed cost k_N, let (r_N^l, q_N^l) and (r_N^u, q_N^u) be the optimal solutions for the lower-bound system $RQ(1, k_N, h_N, \tilde{D}_N, b)$ and the upper-bound system $RQ(1, k_N, \sum_{i=1}^N h_i, \tilde{D}_N, b)$, respectively. One might expect that r_N^* is bounded by (r_N^l, r_N^u) and q_N^* by (q_N^l, q_N^u). The conjecture for q_N^* is not true. For instance, consider a four-stage system with $L_j = 0.25$, $j \leqslant 4$, $h_j = 0.25$, $j \leqslant 3$, $h_4 = 2.5$, $b = 9$, $k = 20$, $\lambda = 16$. The solutions for (r_4^*, q_4^*), (r_4^l, q_4^l), and (r_4^u, q_4^u) are $(13, 12)$, $(13, 11)$, and $(14, 11)$, respectively.

Our findings for this model are shown below.

Theorem 9. *For the serial system $RQ(N, k_N, (h_i, D_i)_{i=1}^N, b)$, we have*

(1) $r_N^l \leqslant r_N^* \leqslant r_N^u$, *and*

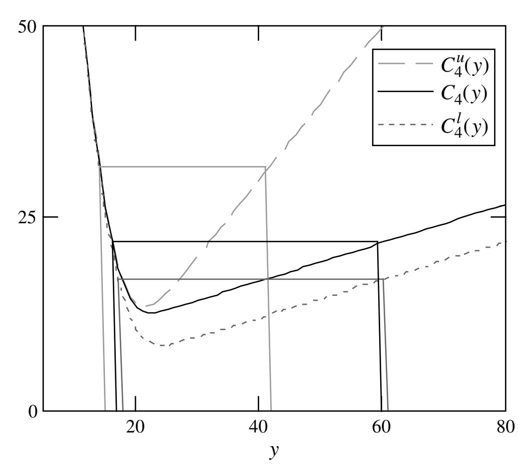
(2) $r_N^l + q_N^l \leqslant r_N^* + q_N^* \leqslant r_N^u + q_N^u$.

Figure 2 illustrates the cost functions for a four-stage system. We only show the cost functions for stage 4 as well as its two single-stage bound systems. It is clear that r_4^* is bounded by r_4^l and r_4^u and $(r_4^* + q_4^*)$ is bounded by $(r_4^l + q_4^l)$ and $(r_4^u + q_4^u)$.

Although the proof of Theorem 9 is quite lengthy, the results are geometrically intuitive. (See Zheng 1992 for a geometric interpretation of the optimality conditions.) This is mostly due to the fact that the upper bound function is "deeper" and the lower-bound function is "shallower" than the original cost function. Thus, to generate the same area λk_N, the "cord" has to move up for the deeper curve, producing lower bounds on r^* and $r^* + q^*$, while the opposite is true for the shallower curve.

Further, let C_N^l be the optimal cost for the lower-bound system $RQ(1, k_N, h_N, \tilde{D}_N, b)$ and C_N^u be the optimal cost for the upper-bound system $RQ(1, k_N, h_1', \tilde{D}_N, b)$. Note that $C_N^u = C_N^u(r_N^l)$ and $C_N^* = C_N(r_N^*)$. Because $r_N^l \leqslant r_N^*$, $C_N^u(r_N^l) \geqslant C_N(r_N^l) \geqslant C_N(r_N^*)$. With the same idea, we have $C_N^l \leqslant C_N^*$.

Figure 2. Illustration of cost functions and bounds at stage 4 with $L_j = 0.25$, $h_j = 0.25\ \forall\, j$, $\lambda = 16$, $b = 9$, $k = 20$.



Proposition 10. $C_N^l \leqslant C_N^* \leqslant C_N^u$.

Single-Stage Heuristics. Similar to the first model with fixed-order quantities, r_N^* and s_j^* can be approximated by any of the two general approaches described in §3.2. Because the shape of the lower-bound function $C_N^l(\cdot)$ is more similar to that of $C_N(\cdot)$, with the same k_N, the resulting optimal order quantity q_N^u should be closer to q_N^*. Therefore, we can use q_N^u to approximate q_N^*. Let $(s_1^a, \ldots, s_{N-1}^a, r_N^a, q_N^a)$ denote these approximations, and we term them as the single-stage heuristic (SSH) solutions.

4.2. Closed-Form Bounds and Heuristics

Let $q_d^l = \sqrt{2\lambda k_N/(h_1'\omega_N^l)}$ and $q_d^u = \sqrt{2\lambda k_N/(h_N\omega_N^u)}$ be the optimal order quantities from the deterministic version of the upper-bound and lower-bound systems at stage N, respectively. Suppose that the demand lead time λL_j is large, and we can approximate it as a normal distribution. Thus, $G_N^u(s_N^l) = (b + h_1')\phi(z_N^l)\sqrt{\mathsf{Var}[\tilde{D}_N]}$. Applying the closed-form results in Proposition 1(2), we have:

Corollary 11. $r_{N-}^l \leqslant r_N^* \leqslant r_{N+}^u$, *where* $r_{N-}^l = \mathsf{E}[\tilde{D}_N] - (1-\omega_N^l)q_d^l - G_N^u(s_N^l)/b$, $r_{N+}^u = s_N^u - (1-\omega_N^u)q_d^u$.

Closed-Form Heuristic. Let $(r_N^c, q_N^c, s_{N-1}^c, \ldots, s_1^c)$ be the closed-form heuristic (CFH) solution. Any linear combination of r_{N-}^l and r_{N+}^u can be an approximation for r_N^c. We use q_d^u to approximate q_N^*. Finally, set $s_j^c = s_j^a$ for $j < N$.

4.3. Numerical Study

We now test the effectiveness of the heuristic solutions through a numerical study. We again use the simple average as the heuristic solutions, i.e., $r_N^a = \langle (r_N^l + r_N^u)/2\rangle$, $s_j^a = \langle (s_j^l + s_j^u)/2\rangle$, $j < N$, and $r_N^c = \lceil (r_{N-}^l + r_{N+}^u)/2\rceil$, $s_j^c = s_j^a$, $j < N$. Also, let $q_N^c = \lceil q_d^u \rceil$. Here, $\lceil\cdot\rceil$ represents the round-up operator. Because the effectiveness of the bounds and approximations for the base-stock levels has been tested and reported in Shang and Song (2003), we focus the numerical study on stage N.

We perform 160 instances with the same holding cost and lead time structures as in the first model: linear, affine, kink, and jump. For each combination of holding cost and lead time forms, we test $k = 2^i$, $i = 0, 1, \ldots, 9$. The rest of the parameters are $\lambda = 32$ and $b = 39$. Table 2 is the summary. The first two columns show the average and maximum percentage cost error for both heuristics; the third and fourth columns report the average and maximum spread of bounds compared to the optimal solution; the fifth and sixth columns report the average and maximum gap between the approximation and the optimal solution for r_4. The last two columns report the same information for q_4. These statistics indicate that the heuristics perform effectively. We may have big gaps for the solution bounds in some cases, but the optimal reorder point tends to be located in the middle of the bounds. Using the optimal order quantity from the lower-bound system, q_4^a, is near optimal. Also, k_N does not seem to affect the quality of the heuristics. For instance, the two biggest errors for SSH in the group of linear holding cost and linear lead time structure are 0.07% and 0.10%, which correspond to $k_N = 1$ and $k_N = 64$, respectively. Similar observations apply to CFH.

We may use different weights on the SSH solutions to improve its performance slightly. For instance, consider the holding-cost weighted average approach, where $r_N^a = \langle \gamma r_N^l + (1-\gamma) r_N^u \rangle$ and $\gamma = h_N/(h_N + h_1')$. (The logic behind this approach is that r_N^* is closer to r_N^u due to the similar shape between $C_N(\cdot)$ and $C_N^l(\cdot)$; see Figure 2. Thus, with fixed k_N, the gap between r_N^* and r_N^u is smaller than that between r_N^* and r_N^l.) The average (maximum) errors for the 160 instances are 0.03% (0.28%), which is slightly better than the simple average approximation.

Table 2. Summary for Model II.

	Cost		Spread of bounds		Solution: r_4		Solution: q_4	
(%)	Avg.	Max.	Avg.	Max.	Avg.	Max.	Avg.	Max.
SSH	0.04	0.22	6.91	17.65	1.64	8.82	0.86	7.14
CFH	0.49	2.49	41.35	60.11	7.13	21.43	3.90	25.00

4.4. Parametric Analysis and Qualitative Insights

The accuracy of the heuristic solutions suggests that we can study the behavior of the optimal policies through parametric analysis of the bounds. We focus on stage N and the optimal system cost only. Proposition 12 provides a summary of the results; the proof is omitted.

Proposition 12. (1) (a) *As* h_j, $j < N$, *increases,* q_N^c *and* r_{N+}^u *remain unchanged, and* r_{N-}^l *and* r_N^c *decrease.* (b) *As* h_N *increases,* q_N^c, r_{N-}^l, r_{N+}^u, *and* r_N^c *decrease.*

(2) *As* b *increases,* q_N^c *decreases, but* r_{N-}^l, r_{N+}^u, *and* r_N^c *increase.*

(3) *As* k_N *increases,* q_N^c *increases, but* r_{N-}^l, r_{N+}^u *and* r_N^c *decrease.*

(4) (a) *Both* r_{N-}^l *and* r_{N+}^u *are independent of the distribution of* $(L_1, \ldots, L_N)$ *as long as* $\tilde{L}_N = \sum_{i=1}^N L_i$ *is fixed.* (b) *If* $\tilde{L}_N$ *increases,* q_N^c *remains unchanged, but* r_{N-}^l, r_{N+}^u, *and* r_N^c *increase.*

Proposition 12(2) suggests that increasing backorder cost rate b leads to a higher optimal reorder point, which in turns, leads to a smaller batch size. From (4)(b), a change in system lead time $\tilde{L}_N$ does not affect q_N^c. This suggests that the optimal order quantity q_N^* is insensitive to $\tilde{L}_N$. These results are consistent with the numerical observation.

We next explore the effects of system parameters on the optimal cost. Applying Proposition 2(2) to the serial model, we have

$$\sqrt{2k_N\lambda h_N\omega_N^u + (b+h_N)^2[\phi(z_N^u)]^2\mathsf{Var}[\tilde{D}_N]} + \tau_N \leqslant C_N^* \leqslant \sqrt{2k_N\lambda h_1'\omega_N^l + bh_1'\mathsf{Var}[\tilde{D}_N]} + \tau_N. \quad (16)$$

As h_j, b, L_j, k, or λ increases, both bounds are nondecreasing. Also, it is not difficult to show that reducing lead time leads to cost savings in both bounds, and the effect is greater for a downstream stage. Numerical observations confirm these properties for the optimal system cost C_N^*.

5. Extension to Compound Poisson Demand

Although this paper focuses on the Poisson demand case, all of our results can be carried over to the compound Poisson demand case. Specifically, when demand follows a compound Poisson process, an echelon (r, nq) policy is optimal for the first model and near optimal for the second model; see Chen (1998, 2000). Let μ and σ^2 be the mean and variance of the demand size Z. All we need to change in our formulas is to replace λ with $\lambda\mu$, $\mathsf{E}[D]$ with $\lambda\mu L$, $\mathsf{Var}[D]$ with $\lambda(\mu^2+\sigma^2)L$, $\mathsf{E}[\tilde{D}_j]$ with $\lambda\mu\tilde{L}_j$, and $\mathsf{Var}[\tilde{D}_j]$ with $\lambda(\mu^2+\sigma^2)\tilde{L}_j$, wherever applicable. Also, our results can be applied to periodic review systems with i.i.d. demand across different time periods (see Chen and Zheng 1994a).

We conduct a numerical study to examine the effectiveness of the heuristics. We assume that the demand size Z follows a geometric distribution, i.e., $\mathsf{P}(Z=x) = (1-\beta)^{x-1}\beta$, $x=1, 2, \ldots, 0<\beta\leqslant 1$. Thus, $\mathsf{E}[Z]=\mu=1/\beta$ and $\mathsf{Var}[Z]=\sigma^2=(1-\beta)/\beta^2$. It can be easily verified that $\mathsf{E}[\tilde{D}_j]=\lambda\tilde{L}_j/\beta$ and $\mathsf{Var}[\tilde{D}_j]=\lambda(2-\beta)\tilde{L}_j/\beta^2$.

For the first model $R(N, (q_i, h_i, D_i)_{i=1}^N, b)$, we consider the same 160 instances in §3.4 with compound Poisson demand. We assume that $\beta=0.5$ and $\lambda=16$ in these instances. The average (maximum) percentage error for SSH is 0.16% (0.79%) and for CFH is 1.01% (2.51%). For the second model $RQ(N, k, (h_i, D_i)_{i=1}^N, b)$, we consider the same 160 instances in §4.3. The average (maximum) percentage error for SSH is 0.06% (0.35%) and for CFH is 0.79% (8.53%). The numerical study suggests that both heuristics do not perform significantly worse due to a larger variance of demand. Thus, we can use these heuristic solutions to perform sensitivity analysis.

Some qualitative properties on r_N^*, q_N^*, and the optimal cost related to μ and σ for the compound Poisson demand case are summarized below. The proof is omitted.

Proposition 13. (1) *In $R(N, (q_i, h_i, D_i)_{i=1}^N, b)$, all properties in Proposition 7 hold. In addition, for all j, r_j^l, r_j^u, r_{j-}^l, r_{j+}^u, $C_N^l(r_N^u)$, and $C_N^u(r_N^l)$ increase in μ; r_{j+}^u, $C_N^l(r_N^u)$, and $C_N^u(r_N^l)$ increase in σ, but r_{j-}^l is nonincreasing in σ.*

(2) *In $RQ(N, k, (h_i, D_i)_{i=1}^N, b)$, all properties in Proposition 12 hold. In addition, q_N^c, r_{N-}^l, r_{N+}^u, and the cost bounds in* (16) *increase in μ; the cost bounds increase in σ; r_{N+}^u increases in σ, but r_{N-}^l decreases in σ.*

Proposition 13 suggests that the optimal costs in both models increase in μ and σ. The optimal reorder points tend to increase in μ but not necessarily increase in σ. This finding is consistent with that in Song (1994) for single-stage base-stock models. Finally, in the second model, the optimal order quantity tends to increase in μ. Although the impact of σ on q_N^c is unclear, we can still expect the optimal order quantity to increase in σ due to the bounds in Gallego (1998). We refer the reader to Zipkin (2000, p. 218) for a discussion of qualitative sensitivity analysis for order quantities.

Remark. The optimal reorder points are not necessarily bounded by the closed-form bounds when the lead time demand is small. Nevertheless, in our numerical study with Poisson demand, the bounds hold in all cases, even though mean lead time demand can be as small as six. When demand is compound Poisson with mean lead time demand six, we do observe instances in which those bounds do not hold. Although the average percentage error of CFH in the compound Poisson demand case is higher, the increased error is not significant.

6. Concluding Remarks

In this paper, we have studied two serial supply chain models for which the optimal policies are echelon (r, q) policies. We have developed effective single-stage and closed-form bounds and approximations for the optimal policy parameters. We have also used these results to conduct sensitivity analysis to gain insights into how system parameters affect performance.

Our analysis enhances our understanding of these two basic multistage models with batch ordering. In particular, we found that, first, under optimal policy, each echelon behaves similarly as a single-stage (r, q) system, regardless of the detailed dynamics of the downstream stages caused by batch ordering; computing a good policy for such systems is almost as simple as that for single-stage systems. Second, in the model with fixed base-order quantities, the optimal reorder points behave similarly as the optimal base-stock levels in serial base-stock systems; the order quantity at a downstream stage may affect the optimal cost dramatically. Third, in the model with fixed cost for external orders, the downstream parameters have little effect on the optimal order quantity at the upmost stage N; the optimal reorder point at stage N, however, tends to decrease as any of the echelon holding costs increases. Fourth, for both models, reducing lead time at a downstream stage offers bigger benefits than at an upstream stage. Finally, the lower-bound system at stage N in both models approximates the exact system well. We expect this result will pave the way toward further development of simple solutions for more general systems with economies of scale.

Appendix

Proof of Proposition 1. We first show Part (2). We introduce a modified deterministic model whose cost function $C_+(r, q)$ is formulated by replacing $G(\cdot)$ in (1) with $G_+(\cdot)$, where $G_+(y) = G(s^*) + h(y - s^*)^+ + b(y - s^*)^-$. In this model, the optimal order quantity is q_d. Let $r_+(x)$ denote the optimal reorder point when the order quantity is x in the modified deterministic model. Thus, the optimal reorder point is $r_+ = r_+(q_d) = s^* - (1-\omega)q_d$ and the optimal cost is $c_+ = c_d + G(s^*)$.

We show $r^* \leqslant r_+$. Let $r(x)$ be the optimal reorder point when the order quantity is x in the $RQ(1, k, h, D, b)$ system. Then, $r'(q) = G'(r(q)+q)/(G'(r(q)) - G'(r(q)+q))$, which implies

$$-1 \leqslant r'(q) \leqslant \frac{-h}{b+h} = r'_+(q). \tag{A1}$$

Thus, $r(q_d) = s^* + \int_0^{q_d} r'(q)\,dq \leqslant s^* + \int_0^{q_d} r'_+(q)\,dq = r_+(q_d)$. Because $r(q)$ is decreasing in q, $r(q_d) \geqslant r(q^*)$. Together, we have $r(q^*) \leqslant r(q_d) \leqslant r_+(q_d)$ or $r^* \leqslant r_+$.

We next show $r_- \leqslant r^*$. Set $H^+ = c_+ - G(s^*)$ and $H^* = G(r^*) - G(s^*)$. Thus, $H^+ \geqslant H^*$. We redefine $r(\cdot)$ as the reorder point where the difference between $G(r(x))$ and $G(s^*)$ is x. In other words, $r(H^*) = r^*$. Note that $r(H^+) \geqslant r_+ - (s^* - \mathsf{E}[D] + G(s^*)/b)$. However, because $H^* \leqslant H^+$, $r^* = r(H^*) \geqslant r(H^+) \geqslant r_+ - (s^* - \mathsf{E}[D] + G(s^*)/b) = r_-$.

For Part (1), because $r^* \geqslant \mathsf{E}[D] - (1-\omega)q_d - G(s^*)/b$ and $q > q_d$ (Zheng 1992), we have $r^* \geqslant r_-$. The upper bound is a direct result of (A1).

Proof of Proposition 2. We only need to prove Part (1). Let k' be the corresponding fixed-order cost such that the optimal solutions are (r^*, q). Consider the same modified deterministic model in the proof of Proposition 1. Let $q_d(k')$ be the optimal order quantity for this deterministic model. Thus, $q \geqslant q_d(k')$ and $C(r^*, q) \leqslant G_+(s^*) + h\omega q_d(k') \leqslant G(s^*) + h\omega q$. On the other hand, consider the EOQ model whose cost function $C_-(r, q)$ is formulated by replacing $G(\cdot)$ in (1) with $G_-(y) \stackrel{\text{def}}{=} h(y - \mathsf{E}[D])^+ + b(y - \mathsf{E}[D])^-$. Because $G(\cdot) \geqslant G_-(\cdot)$, $G(r^*)$ must be higher than $qh\omega$, the optimal cost for this deterministic model with q. Finally, because $C(r^*, q) \geqslant G(s^*)$, we have $C(r^*, q) \geqslant \max\{G(s^*), qh\omega\}$.

Proof of Theorem 4. Note that (2) and (3) are implied by (1), and (5) is implied by (4). To prove (1), we first review a result developed by Shang and Song (2003). When $q_j = 1$, $C_j(y)$ reduces to $C_j^1(y)$, where

$$C_1^1(y) = C_1(y) = \mathsf{E}[h_1(y - D_1) + (b + h'_1)(y - D_1)^-],$$

$$C_j^1(y) = \mathsf{E}[h_j(y - D_j) + C_{j-1}^1(\min\{s_{j-1}^*, y - D_j\})], \quad j = 2, \ldots, N,$$

and $C_j^l(y) \leqslant C_j^1(y) \leqslant C_j^u(y)$, $j = 1, \ldots, N$.

We first show $C_j^l(y) \leqslant C_j(y)$ by induction. It is sufficient to show that $C_j^1(y) \leqslant C_j(y)$. When $k = 1$, $C_1^1(y) = C_1(y)$. Suppose that $k = j-1$ is true, i.e., $C_{j-1}^1(y) \leqslant C_{j-1}(y)$. When $k = j$,

$$\begin{aligned} C_j^1(y) &= \mathsf{E}[h_j(y - D_j) + C_{j-1}^1(\min\{s_{j-1}^*, y - D_j\})] \\ &\leqslant \mathsf{E}[h_j(y - D_j) + C_{j-1}(\min\{s_{j-1}^*, y - D_j\})] \\ &\qquad \text{(by the induction assumption)} \\ &\leqslant \mathsf{E}[h_j(y - D_j) + C_{j-1}(O_{j-1}[y - D_j])] = C_j(y). \end{aligned}$$

Thus, $C_j^1(y) \leqslant C_j(y)$ and $C_j^l(y) \leqslant C_j(y)$.

We next show $C_j(y) \leqslant C_j^u(y)$. When $k = 1$, $C_1(y) = C_1^u(y)$. Suppose that $k = j - 1$ is true, i.e., $C_{j-1}(y) \leqslant C_{j-1}^u(y)$. When $k = j$, we have

$$\begin{aligned} C_j(y) &= \mathsf{E}[h_j(y - D_j) + C_{j-1}(O_{j-1}[y - D_j])] \\ &\leqslant \mathsf{E}[h_j(y - D_j) + C_{j-1}^u(O_{j-1}[y - D_j])] \\ &\qquad \text{(by the induction assumption)} \\ &\leqslant \mathsf{E}[h_j(y - D_j) + C_{j-1}^u(y - D_j)] \\ &= \mathsf{E}[h_j(y - D_j) + h[1, j-1](y - D_j - \tilde{D}_{j-1}) \\ &\qquad + (b + h'_1)(y - D_j - \tilde{D}_{j-1})^-] = C_j^u(y). \end{aligned}$$

Thus, $C_j(y) \leqslant C_j^u(y)$.

We now show (4) by induction. $\Delta\bar{C}_1(y) = \Delta\bar{C}_1^l(y)$. By definition,

$$\begin{aligned} \Delta\bar{C}_j(y) &= \sum_{z=0}^{n_{j-1}-1} [h_j q_{j-1} + \Delta\mathsf{E}\bar{C}_{j-1}(\min\{r_{j-1}^*, y + zq_{j-1} - D_j\})] \\ &\stackrel{\text{def}}{=} \sum_{z=0}^{n_{j-1}-1} \Delta H_j(z). \end{aligned}$$

Suppose that $j-1$ is true, i.e., $\Delta\bar{C}_{j-1}(y) \geqslant \Delta\bar{C}_{j-1}^l(y)$. Conditioning on $D_j = d_j$, if $y + zq_{j-1} - d_j \geqslant r_{j-1}^*$, then

$$\begin{aligned} \Delta H_j(z) &= h_j q_{j-1} + \Delta\bar{C}_{j-1}(r_{j-1}^*) \\ &\geqslant h_j q_{j-1} - (b + h'_j)[\mathsf{P}(\tilde{D}_j > y + zq_{j-1} - d_j) \\ &\qquad + \cdots + \mathsf{P}(\tilde{D}_j > y + (z+1)q_{j-1} - 1 - d_j)] \\ &= \Delta\bar{C}_j^l(y + zq_{j-1} \mid d_j). \end{aligned}$$

If $y + zq_{j-1} - d_j < r_{j-1}^*$, then

$$\begin{aligned} \Delta H_j(z) &= h_j q_{j-1} + \Delta\bar{C}_{j-1}(y + zq_{j-1} - d_j) \\ &\geqslant h_j q_{j-1} + \Delta\bar{C}_{j-1}^l(y + zq_{j-1} - d_j) \\ &\qquad \text{(by the induction assumption)} \\ &= h_j q_{j-1} + h_{j-1} q_{j-1} - (b + h'_{j-1}) \\ &\qquad \cdot [\mathsf{P}(\tilde{D}_j > y + zq_{j-1} - d_j) \\ &\qquad + \cdots + \mathsf{P}(\tilde{D}_j > y + (z+1)q_{j-1} - 1 - d_j)] \\ &\geqslant \Delta\bar{C}_j^l(y + zq_{j-1} \mid d_j). \end{aligned}$$

Thus, $\Delta H_j(z) \geqslant \Delta\bar{C}_j^l(y + zq_{j-1})$. Consequently,

$$\begin{aligned} \Delta\bar{C}_j(y) &= \sum_{z=0}^{n_{j-1}-1} \Delta H_j(z) \geqslant \sum_{z=0}^{n_{j-1}-1} \Delta\bar{C}_j^l(y + zq_{j-1}) \\ &= \sum_{x=1}^{q_j} \Delta C_j^l(y + x) = \Delta\bar{C}_j^l(y). \end{aligned}$$

Finally, we show $\Delta C_j^u(y) \geqslant \Delta C_j(y)$.

$$\begin{aligned}\Delta\bar{C}_j(y) &= \sum_{z=0}^{n_{j-1}-1}\sum_{x=1}^{q_{j-1}}(h_j+\Delta \mathsf{E} C_{j-1}(O_{j-1}[y+x+zq_{j-1}-D_j]))\\ &= \sum_{z=0}^{n_{j-1}-1}\sum_{x=1}^{q_{j-1}}\Delta C_j(y+x+zq_{j-1})\\ &\leqslant \sum_{z=0}^{n_{j-1}-1}\sum_{x=1}^{q_{j-1}}\Delta C_j^u(y+x+zq_{j-1}) = \Delta\bar{C}_j^u(y).\end{aligned}$$

This completes the proof.

Proof of Theorem 9. We only need to show the result for stage N. For ease of presentation, we treat the demand and decision variables as continuous variables. (The discrete case can be shown similarly with additional technicality because the optimality condition involves two inequalities.)

When demand is continuous, (15) becomes

$$\min_{r_N,\,q_N} C(r_N, q_N) = \frac{k\lambda + \int_{r_N}^{r_N+q_N} C_N(y)\,dy}{q_N}.$$

Note that $C_N^{l'}(y) \leqslant C_N'(y) \leqslant C_N^{u'}(y)$, $C_N^l(y) \leqslant C_N(y) \leqslant C_N^u(y)$, and $s_N^l \leqslant s_N^* \leqslant s_N^u$ (Shang and Song 2003). We first show $r_N^l \leqslant r_N^*$ in Part (1) by contradiction.

Suppose that $r_N^l > r_N^*$ is true; then $C_N(r_N^*) > C_N(r_N^l)$. Note that $C_N^{u'}(y) \geqslant C_N'(y)$ for all $y \geqslant 0$. Also, $C_N^u(s_N^l) = C_N^u(r_N^l) + \int_{r_N^l}^{s_N^l} C_N^{u'}(y)\,dy$ and $C_N(s_N^*) = C_N(r_N^l) + \int_{r_N^l}^{s_N^*} C_N'(y)\,dy$. Because $\int_{r_N^l}^{s_N^*} C_N'(y)\,dy \leqslant \int_{r_N^l}^{s_N^l} C_N'(y)\,dy \leqslant \int_{r_N^l}^{s_N^l} C_N^{u'}(y)\,dy < 0$, we have

$$C_N(r_N^*) - C_N(s_N^*) > C_N(r_N^l) - C_N(s_N^*) \geqslant C_N^u(r_N^l) - C_N^u(s_N^l).$$

At optimality, $C_N(r_N^*) = C_N(r_N^* + q_N^*)$ and $C_N^u(r_N^l) = C_N^u(r_N^l + q_N^l)$. Therefore, $C_N(r_N^* + q_N^*) - C_N(s_N^*) > C_N^u(r_N^l + q_N^l) - C_N^u(s_N^l)$, or equivalently,

$$\int_{s_N^*}^{r_N^*+q_N^*} C_N'(y)\,dy > \int_{s_N^l}^{r_N^l+q_N^l} C_N^{u'}(y)\,dy.$$

Note that $C_N^{u'}(y) \geqslant C_N'(y)$ and $s_N^* \geqslant s_N^l$. As a result, $r_N^* + q_N^* > r_N^l + q_N^l$. That is, we have shown that if $r_N^* < r_N^l$ is true, then $r_N^* + q_N^*$ must be greater than $r_N^l + q_N^l$.

Now we are ready to use this result to show the contradiction of the assumption. From Zheng (1992), we have

$$\begin{aligned}k\lambda &= q_N^l C_N^u(r_N^l) - \int_{r_N^l}^{r_N^l+q_N^l} C_N^u(y)\,dy\\ &= q_N^l C_N(r_N^l)\\ &\quad - \int_{r_N^l}^{r_N^l+q_N^l}[C_N^u(y) + C_N(r_N^l) - C_N^u(r_N^l)]\,dy \qquad \text{(A2)}\\ &\leqslant q_N^l C_N(r_N^l)\\ &\quad - \int_{r_N^l}^{r_N^l+q_N^l}[C_N^u(y) + C_N(y) - C_N^u(y)]\,dy \qquad \text{(A3)}\\ &< q_N^l C_N(r_N^*) - \int_{r_N^l}^{r_N^l+q_N^l} C_N(y)\,dy\\ &= q_N^* C_N(r_N^*) - \int_{r_N^*}^{r_N^*+q_N^*} C_N(y)\,dy + (q_N^l - q_N^*)C_N(r_N^*)\\ &\quad + \int_{r_N^*}^{r_N^l} C_N(y)\,dy + \int_{r_N^l+q_N^l}^{r_N^*+q_N^*} C_N(y)\,dy\\ &= q_N^* C_N(r_N^*) - \int_{r_N^*}^{r_N^*+q_N^*} C_N(y)\,dy\\ &\quad - \left[(r_N^* + q_N^* - r_N^l - q_N^l)C_N(r_N^*) - \int_{r_N^l+q_N^l}^{r_N^*+q_N^*} C_N(y)\,dy\right]\\ &\quad - \left[(r_N^l - r_N^*)C_N(r_N^*) - \int_{r_N^*}^{r_N^l} C_N(y)\,dy\right]\\ &= q_N^* C_N(r_N^*) - \int_{r_N^*}^{r_N^*+q_N^*} C_N(y)\,dy\\ &\quad - \int_{r_N^l+q_N^l}^{r_N^*+q_N^*}[C_N(r_N^*) - C_N(y)]\,dy\\ &\quad - \int_{r_N^*}^{r_N^l}[C_N(r_N^*) - C_N(y)]\,dy \qquad \text{(A4)}\\ &\leqslant q_N^* C_N(r_N^*) - \int_{r_N^*}^{r_N^*+q_N^*} C_N(y)\,dy \qquad \text{(A5)}\\ &= k\lambda.\end{aligned}$$

The inequality between (A2) and (A3) holds because $0 > C_N(r_N^l) - C_N^u(r_N^l) \geqslant C_N(y) - C_N^u(y)$ for $y \geqslant r_N^l$. Also, the inequality between (A4) and (A5) holds because $C_N(r_N^*) \geqslant C_N(y)$ as $r_N^* \leqslant y \leqslant r_N^* + q_N^*$, $r_N^l > r_N^*$, and $r_N^* + q_N^* > r_N^l + q_N^l$.

Thus, we obtain $k\lambda < k\lambda$, a contradiction. Therefore, we must have $r_N^l \leqslant r_N^*$. A similar argument can be applied to show that $r_N^* \leqslant r_N^u$. This is because C_N and C_N^l have the same symmetric structure as C_N^u and C_N. Together, we have $r_N^l \leqslant r_N^* \leqslant r_N^u$.

We now proceed to show Part (2). Again, we prove $r_N^l + q_N^l \leqslant r_N^* + q_N^*$ by contradiction. Suppose that $r_N^l + q_N^l > r_N^* + q_N^*$ is true. Note that $C_N^u(r_N^l + q_N^l) - C_N^u(s_N^l) = \int_{s_N^l}^{r_N^l+q_N^l} C_N^{u'}(y)\,dy$ and $C_N(r_N^* + q_N^*) - C_N(s_N^*) = \int_{s_N^*}^{r_N^*+q_N^*} C_N'(y)\,dy$. Because $C_N^{u'}(y) \geqslant C_N'(y)$ for $y \geqslant 0$ and $s_N^* \geqslant s_N^l$, therefore $C_N^u(r_N^l + q_N^l) - C_N^u(s_N^l) > C_N(r_N^* + q_N^*) - C_N(s_N^*)$. Because $C_N^u(r_N^l + q_N^l) = C_N^u(r_N^l)$ and $C_N(r_N^* + q_N^*) = C_N(r_N^*)$, we have $C_N^u(r_N^l) - C_N^u(s_N^l) > C_N(r_N^*) - C_N(s_N^*)$, or equivalently,

$$-\int_{r_N^l}^{s_N^l} C_N^{u'}(y)\,dy > -\int_{r_N^*}^{s_N^*} C_N'(y)\,dy.$$

Because $s_N^* \geqslant s_N^l$ and $0 > C_N^{u'}(y) > C_N'(y)$, therefore we have $r_N^* > r_N^l$.

Suppose that the assumption is true then,

$$
\begin{aligned}
k\lambda &= q_N^* C_N(r_N^*) - \int_{r_N^*}^{r_N^*+q_N^*} C_N(y)\,dy \\
&= q_N^*[C_N(r_N^*) + C_N^u(r_N^*+q_N^*) - C_N(r_N^*+q_N^*)] \\
&\quad - \int_{r_N^*}^{r_N^*+q_N^*} [C_N(y) + C_N^u(r_N^*+q_N^*) \\
&\qquad\qquad - C_N(r_N^*+q_N^*)]\,dy \qquad \text{(A6)} \\
&\leqslant q_N^* C_N^u(r_N^*+q_N^*) \\
&\quad - \int_{r_N^*}^{r_N^*+q_N^*} [C_N(y) + C_N^u(y) - C_N(y)]\,dy \qquad \text{(A7)} \\
&< q_N^* C_N^u(r_N^l+q_N^l) - \int_{r_N^*}^{r_N^*+q_N^*} C_N^u(y)\,dy \\
&= q_N^l C_N^u(r_N^l+q_N^l) - \int_{r_N^l}^{r_N^l+q_N^l} C_N^u(y)\,dy \\
&\quad - (r_N^* - r_N^l) C_N^u(r_N^l+q_N^l) \\
&\quad - (r_N^l + q_N^l - r_N^* - q_N^*) C_N^u(r_N^l+q_N^l) \\
&\quad + \int_{r_N^*}^{r_N^l} C_N^u(y)\,dy + \int_{r_N^*+q_N^*}^{r_N^l+q_N^l} C_N^u(y)\,dy \\
&= q_N^l C_N^u(r_N^l+q_N^l) - \int_{r_N^l}^{r_N^l+q_N^l} C_N^u(y)\,dy \\
&\quad - \int_{r_N^l}^{r_N^*} [C_N^u(r_N^l+q_N^l) - C_N^u(y)]\,dy \\
&\quad - \int_{r_N^*+q_N^*}^{r_N^l+q_N^l} [C_N^u(r_N^l+q_N^l) - C_N^u(y)]\,dy \qquad \text{(A8)} \\
&< q_N^l C_N^u(r_N^l+q_N^l) - \int_{r_N^l}^{r_N^l+q_N^l} C_N^u(y)\,dy \qquad \text{(A9)} \\
&= k\lambda.
\end{aligned}
$$

The inequality between (A6) and (A7) holds because $C_N^u(r_N^* + q_N^*) - C_N(r_N^* + q_N^*) > C_N^u(y) - C_N(y)$ for $y \leqslant r_N^* + q_N^*$. The inequality between (A8) and (A9) holds because $C_N^u(r_N^l + q_N^l) > C_N^u(y)$ for $r_N^l \leqslant y \leqslant r_N^l + q_N^l$, $r_N^* > r_N^l$, and $r_N^l + q_N^l > r_N^* + q_N^*$.

Thus, a contradiction occurs so that $r_N^l + q_N^l > r_N^* + q_N^*$ is not true. As a result, $r_N^l + q_N^l \leqslant r_N^* + q_N^*$. A similar proof can be applied to show $r_N^* + q_N^* \leqslant r_N^u + q_N^u$.

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