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An Inventory System with Cash Flows [1]

Consider a firm facing nonstationary random demand in a finite horizon. At the beginning of each period, an order is placed to meet the uncertain demand. The objective is to maximize the expected net worth (equity) at the end of the horizon. For simplicity, let us assume the firm has no investment activities other than investing in inventory and in the money markets with the cash it accumulates. Also, we do not consider long-term assets and liabilities. Thus, the objective is equivalent to maximizing the expected working capital at the end of the horizon. Different objectives have been proposed in the textbooks. For example, Baye and Prince (2014) define that the value of a firm is the present value of the firm's current and future profits. As we shall see later, this objective is equivalent to maximizing the expected terminal working capital we assume here. In some corporate finance textbooks, the value of a firm is defined as the present value of the total dividend payments. Notice that the dividend payment cycle for most firms is on a quarterly or bi-annual basis, whereas the inventory order decision is mostly on a daily or weekly basis. Thus, our objective of maximizing the net worth of the firm does not contradict the latter perspective as the firm can maximize its net worth (equity) for better utilizing its dividend policy (i.e., either paying dividends with cash or keeping cash as retained earnings) at the end of the planning horizon.

Without loss of generality, we assume that the order lead time is zero. Define the following parameters for the system.

$$\begin{aligned}
h_p &= \text{physical holding cost rate (\$/unit/period);}\\
b_p &= \text{physical backorder cost rate (\$/unit/period);}\\
r &= \text{interest return rate for investing in the money market;}\\
d &= \text{borrowing rate from the capital market, where } d \geq r;\\
p &= \text{unit selling price;}\\
c &= \text{unit purchase cost.}
\end{aligned}$$

Here, the physical holding cost rate h_p refers to the costs related to storing and maintaining inventory, insurances, shrinkage, etc. average out total units sold over a certain period of time. There are more discussions on assessing the costs later in this note. For now, let's view h_p as the cost rate that does *not* include the financial opportunity cost due to holding inventory. The physical backorder cost rate b_p should be viewed the same way – it is the tangible, monetary penalty costs related to backlogging, e.g., expediting delivery costs.

We count the time forward, i.e., $t = 1, 2, ..., t, t+1, ...T$, where T is the end of the horizon. Let D_t is the demand occurred in period t. To examine the system dynamics, we introduce the following state variables:

$$x_t = \text{net inventory level at the beginning of period } t;$$

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$$w_t' = \text{net cash level at the beginning of period } t;$$
$$w_t = \text{working capital at the beginning of period } t \ = w_t' + cx_t.$$

Let y_t is the inventory position after ordering, and the order quantity is $(y_t - x_t)$. Define the inventory-related cost and the cash-related gain in period t:

$$\begin{aligned} G_t(y_t) &= \mathbf{E}[h_p(y_t - D_t)^+ + b_p(y - D_t)^-], \\ R_t(y_t) &= r(w_t' - c(y_t - x_t))^+ - d(w_t' - c(y_t - x_t))^-, \end{aligned}$$

where $x^+ = \max\{x, 0\}$, $x^- = -\min\{x, 0\}$, and $\mathbf{E}[\cdot]$ is the expected value over the random demand. The first term $(y_t - D_t)^+$ in the G_t function is the on-hand inventory at the end of the period, whereas the second term $(y_t - D_t)^-$ is the backorder level. The G_t function represents the inventory holding and backorder cost, or the inventory-related cost in short in period t. Similarly, the term $(w_t' - c(y_t - x_t))$ in the R_t function is the net cash level after inventory payment. It yields an interest gain r if positive and an interest loss d if negative (representing the borrowing rate from financial institutions in order to pay the ordered inventory). The R_t function is the cash-related gain in period t.

The transitions of the inventory and cash states between periods are as follows:

$$\begin{aligned} x_{t+1} &= y_t - D_t, & (1) \\ w_{t+1}' &= w_t' - c(y_t - x_t) + pD_t + R_t(y_t) - G_t(y_t) \\ &= (1+r)(w_t' - c(y_t - x_t))^+ - (1+d)(w_t' - c(y_t - x_t))^- + pD_t - G_t(y_t). & (2) \end{aligned}$$

Equation (2) states that the next period's net cash is the result of total cash inflows and outflows of the current period. Here, we assume that the customer will pay on order so the revenue is pD_t. If there is debt, i.e., $(w_t' - c(y_t - x_t))$ is negative, it will incur an interest loss and this debt will carry over to the next period.

The net working capital is

$$\begin{aligned} w_{t+1} &= (1+r)(w_t - cy_t)^+ - (1+d)(w_t - cy_t)^- + pD_t - G_t(y_t) + c(y_t - D_t) \\ &= (1+r)w_t + (p-c)D_t - rcy_t - G_t(y_t) - (d-r)(w_t - cy_t)^- & (3) \\ &= w_t + (p-c)D_t - G_t(y_t) + r(w_t - cy_t)^+ - d(w_t - cy_t)^-. & (4) \end{aligned}$$

So far, we have considered a very practical and general environment for a firm that can invest and borrow with different rates.

1.1 Perfect Financial Markets

Let's consider a special case in which the financial market is *perfect*.[2] In our model, this is equivalent to assuming that $r = d$, i.e., the interest return rate is equal to the borrowing rate.

[2] Modigliani and Miller (1958) show that if the financial markets are perfect (no frictions), i.e., absence of taxes, bankruptcy costs, agency costs, asymmetric information, the value of a firm is unaffected by how that

In this case, the firm can freely borrow cash so cash is no longer a concern. This is exactly what the classic inventory model assumes. With this assumption, Equation (4) becomes

$$\begin{aligned} & (1+r)w_t + (p-c)D_t - rcy_t - G_t(y_t) \\ = \; & (p - c(1+r))D_t + (1+r)w_t - (h_p + rc)(y_t - D_t)^+ - (b_p - rc)(y_t - D_t)^-. \end{aligned} \qquad (5)$$

Notice that D_t is an exogenous random variable, and w_t is the initial system state at the beginning of period t. Thus, to maximize the expected working capital in period $t+1$, one only needs to minimize the expected cost in Equation (5), i.e.,

$$\mathbf{E}[(h_p + rc)(y_t - D_t)^+ + (b_p - rc)(y_t - D_t)^-]. \qquad (6)$$

Equation (6) is the single-period inventory-related cost in in the classic inventory model. We have a sound economic meaning for the cost parameters. The holding cost rate h is $(h_p + rc)$, which is the sum of the physical holding cost rate h_p and the opportunity cost of capital rc due to inventory investment. The backorder cost rate b is $(b_p - rc)$, which is the physical backorder cost minus the opportunity cost of capital rc.[3] It can be shown that a base-stock policy is optimal. In the Fuqua teaching note *An Introduction to Inventory Management*, we mention that the holding cost rate h is composed of two parts, the physical holding cost rate and the financial cost of capital due to inventory purchase. The above analysis has demonstrated this point. More specifically, one can view the holding cost rate h and backorder cost rate b as follows:

$$\begin{aligned} h &= h_p + rc; \\ b &= b_p - rc. \end{aligned}$$

If the demand is i.i.d, the optimal base-stock level s^* can be obtained from the well-known optimality equation as shown in the inventory teaching note: Finding s^* such that

$$P(D \le s^*) = \left(\frac{b_p - rc}{b_p + h_p}\right) = \left(\frac{b}{b+h}\right).$$

For simplicity, let's write

$$s^* = F^{-1}\left(\frac{b}{b+h}\right),$$

where F is the cdf of the demand distribution and F^{-1} is the inverse cdf function.[4]

firm is financed. This is often referred to *the capital structure irrelevance principle.* Notice that the value of the firm is often created by the operations activities. In other words, this principle basically suggests that operations decisions may be decoupled from finance decisions. Unfortunately, it is not possible that the markets are of no frictions. In our inventory model, the friction comes from the fact that d is larger than r as the financial institutions have to make a non-zero profit.

[3] It can be shown that the discount rate α is equal to $1/(1+r)$.

[4] If lead time is positive, say L, the demand should be replaced with the total demand during lead time and the review period.

We pose a discussion on estimating the cost parameters in practice here. According to the above analysis, when the financial market is perfect, the inventory holding cost rate is composed of the physical holding cost rate h_p and the financial holding cost rate rc. Recall that r is the interest rate due to the cash investment in the capital market. Broadly speaking, if a firm conducts investments by financing through debt and equity, the required expected return would be WACC, the weighted average cost of capital. In most OM textbooks, WACC is often recommended to estimate the financial holding cost rate. To estimate the physical holding cost rate, however, is a tough task. The physical holding cost comes from, for example, managing and maintenance expenses, storage, insurances, shrinkage, obsolescence, etc. The list is very long and often business specific. Technically speaking, one should sum up all these costs in a time period and allocate the cost to each inventory unit sold during this period. There are a number of alternative cost accounting systems that can be relevant for some purposes while being inadequate for others. Thus, it is neither always possible nor economical to keep track of all costs, or to split them and allocate them properly. Fortunately, the physical holding cost rate is often small.

As for the backorder rate, this is the penalty cost incurred for an arriving demand that cannot be satisfied immediately due to stock out. Some physical (tangible) penalty costs may occur, e.g., expedited shipping cost, overtime production, etc. Let b_p denote the physical backorder cost. Then, the backorder cost rate is $b = b_p - rc$. This is because a unit of inventory shortage implies that additional cash c was invested in the capital market, gaining the return rate r. Thus, the actual tangible backorder cost rate is b, which is less than b_p. Nevertheless, one should know the backorder cost is beyond the tangible costs. There are many intangible costs involved, such as loss of goodwill, market shrinkage, etc.

1.2 Imperfect Financial Markets

So far, we have explained how the inventory model as well as its corresponding cost parameters are formulated when the financial market is perfect, i.e., $d = r$. Let's turn to a more realistic scenario where the financial market is not perfect, i.e., $d > r$. In this case, Equation (4) suggests that to maximize the expected working capital in period $t+1$, one has to minimize $\mathbf{E}[G_t(y_t)] - R_t(y_t)$, or equivalently,

$$\mathbf{E}[h_p(y_t - D_t)^+ + b_p(y_t - D_t)^-] - r(w_t^{'} - c(y_t - x_t))^+ + d(w_t^{'} - c(y_t - x_t))^-. \qquad (7)$$

One can see that the inventory decision y_t is affected by the cash level $w_t^{'}$ through the last two terms of Equation (7). Thus, the inventory decision cannot be decoupled from the financial flow.

The optimal inventory policy no longer has a simple structure. Luo and Shang (2018) shows that a base-stock policy that depends on the working capital level is near-optimal. More specifically, when demand is i.i.d, define

$$\begin{aligned} \underline{s} &= F^{-1}\left(\frac{b_p - dc}{b_p + h_p}\right) = F^{-1}\left(\frac{b_p - dc}{b + h}\right), \\ s &= F^{-1}\left(\frac{b_p - rc}{b_p + h_p}\right) = F^{-1}\left(\frac{b}{b + h}\right). \end{aligned}$$

Clearly, $\underline{s} \leq s$. The optimal policy is executed as follows. When the working capital level w_t is greater (less, respectively) than cs ($c\underline{s}$, respectively), one should order up to the base stock level s ($\underline{s}$, respectively). If the working capital is between cs and $c\underline{s}$, one should order up to the total working capital level w_t. We refer to this optimal policy as the $(\underline{s}, s)$ policy.

References

1. Baye, M., J. Frince. 2014. *Managerial Economics and Business Strategy.* McGraw-Hill Irwin, New York, NY.

2. Luo, W., K. Shang. 2018. Managing inventory for firms with trade credit and payment defaults. Forthcoming in *Operations Research.*

3. Modigliani,F., M. Miller. 1958. The cost of capital, corporation finance and the theory of investment. *American Economic Review* 48(3) 261-297.